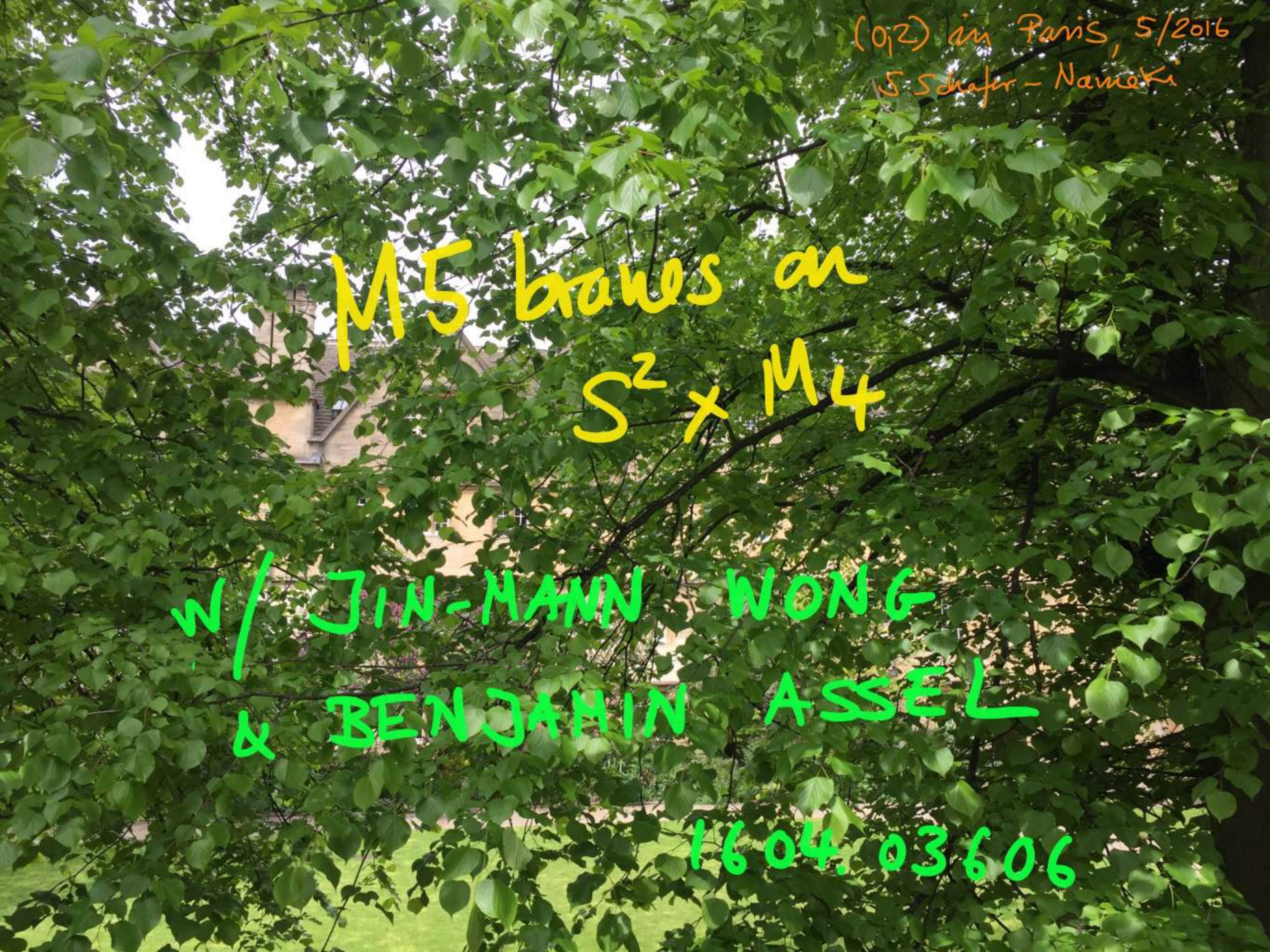

(0,2) in Paris, 5/2016
S. Schafer-Nameki
M5 branes on $S^2 \times M_4$
w/ JIN-MANN WONG
& BENJAMIN ASSEL
1604.03606

$$6 = 4 + 2$$

$$(\neq 2 + 4)$$

PLAN

1. Motivation: Dualities from M5-branes
2. M5s & Topological twists.
3. $6d \longrightarrow 5d$
4. $5d \longrightarrow 4d$

$\left.\begin{matrix} 6d \longrightarrow 5d \\ 5d \longrightarrow 4d \end{matrix}\right\}$ 6d (0,2) on S^2

5. 4d Topological σ-model from M5 on S^2
6. Conclusions.

1 Motivations

$$6d\ (0,2)\ \text{SCFT}\ ,\ SU(N) \begin{array}{l} \xrightarrow{S^1} 5d\ N=2\ \text{SYM} \\ \xrightarrow{T^2} 4d\ N=4\ \text{SYM} \end{array}$$

= theory on N M5-branes

Challenge in studying 6d (0,2):

→ no (known) Lagrangian formulation

→ eoms for abelian theory only.

→ $Sp(4)_R \times SO(6)_L \subseteq OSp(6|4)$

Scalars Φ: $(5,1)$

Selfdual 2-form B: $(1,15)$ ← $H = dB = *H$

Fermions ρ: $(4,\bar{4})$

This lack of understanding has not stopped us from studying dimensional Reductions of the 6d (0,2) theory:

$$6 = (6-p) + p$$

M5 on $S^p \times M_{6-p}$

$\mathrm{Vol}(S^p) \to 0$

$\mathrm{Vol}(M) \to 0$

(6-p) dim
Topological / Conformal Theory
on M_{6-p}

$\longleftrightarrow$

p-dim. $N=2$
SUSY theory on S^p

duality/correspondence

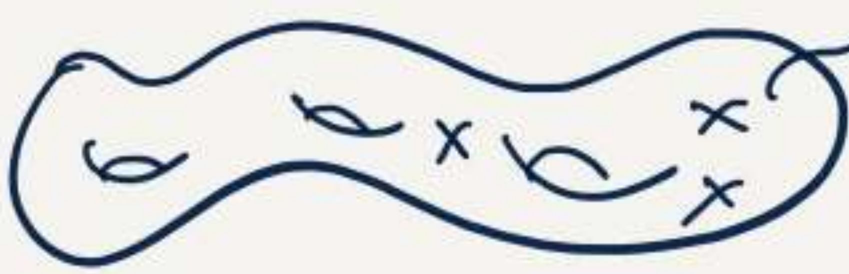
$P=4$: Alday-Gaiotto-Tachikawa Correspondence (AGT)

M_2 = Riemann surface (w/ punctures)

$$\boxed{6d\ (0,2) \text{ on } S^4 \times M_2}$$

$0 \leftarrow \mathrm{Vol}(S^4)$ | $\mathrm{Vol}(M_2) \to 0$

Liouville / Toda theory on M_2	$\mathcal{N}=2$ SUSY class S theory on S^4: $T[M_2]$

* 

* $T\Big[$  $\Big]$ [Gaiotto]

* conformal blocks & correlation fn. on M_2:

* Nekrasov-Pestun partition fn. on S^4:

$$\Big\langle \prod_i V_{m_i} \Big\rangle_{M_2^{\tau}} = \Big| Z_{T[M_2]}[S^4] \Big|^2 (\tau, m_i)$$

hyper.m. masses

couplings

* Mapping class group (pair of pants decomp) = S-duality of class S theory.

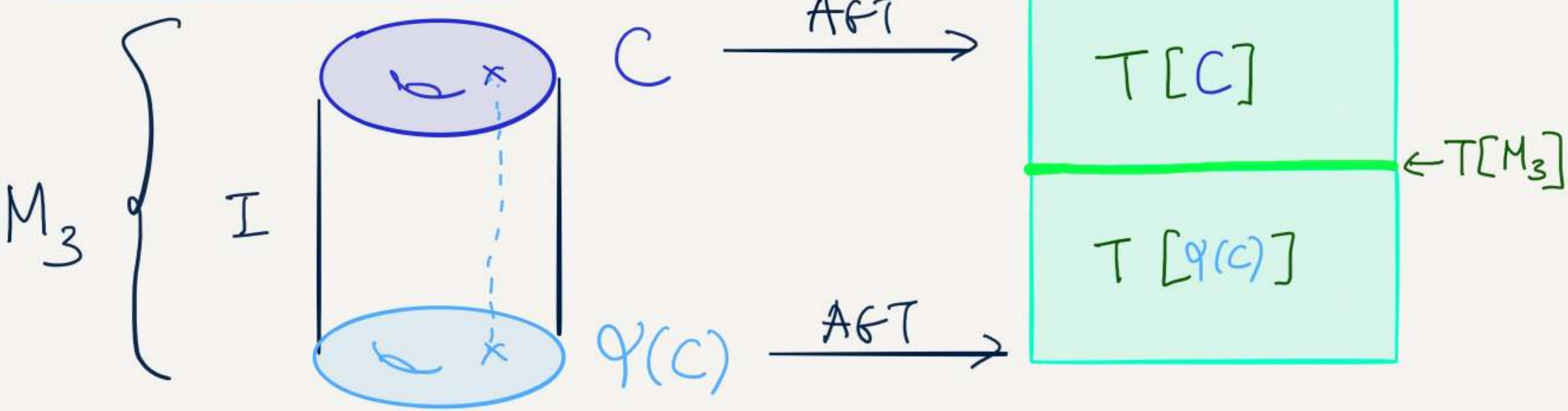
$\boxed{p=3}$

3d-3d duality

[Dimofte Gaiotto Gukov; Terashima, Yamazaki]

Complex Chern Simons on M3	$\longleftrightarrow$	3d $N=2$ SUSY $T[M_3]$
* $SL_2\mathbb{C}$ partition fn.		* Partition fn of $T[M_3]$ on S^3
* change of triangulation		* S-duality

3d-3d from AGT



General lessons:

- AGT & 3d3d correspondences provide surprising relations between

  * $\boxed{\mathcal{N}=2 \text{ SUSY gauge theories}}$
    - partition fn. , index
    - defects, line ops ...
  * $\boxed{\text{Conformal / Topological theories}}$
    - Conformal blocks, partition fn.
    - Verlinde line ops ...

- Definition of $\mathcal{N}=2$ SUSY theories in p dim. via $(6-p)$ dim. manifolds: $T[M_{6-p}]$.

Natural Question: $\boxed{p=2}$

[Assel, SSN, Wang]

Expectation:

$$\begin{tikzcd}[column sep=small]
& \boxed{\text{6d (0,2) on } S^2 \times M_4} \arrow[dl, "\mathrm{Vol}(S^2)\to 0"'] \arrow[dr, "\mathrm{Vol}(M_4)\to 0"] & \\
\boxed{\begin{array}{c}\text{4d Topological Theory}\\ \text{on } M_4\end{array}} & & \boxed{\begin{array}{c}\text{2d (0,2) SUSY}\\ \text{gauge th. } T[M_4] \text{ on } S^2\end{array}}
\end{tikzcd}$$

$\Rightarrow$ "4d-2d correspondence":

New way to define 2d (0,2) theories, & compute via 4d.

Question: - What is the 4d theory?

- What is the dictionary?

② 6d (0,2) on $S^2 \times M_4$: Topological Twists

$$\boxed{Sp(4)_R \times SO(6)_L}$$

on $S^2 \times M_4$

(0,2) Twist

[Assel, SSN, Wang]

$\boxed{SU(2)_R} \times SO(2)_R$

$SO(2)_L \times \boxed{SU(2)_\ell} \times SU(2)_r$

$$\boxed{T_{twist} = \tfrac{1}{2}(T_R + T_\ell)}$$

Similar to 4d Vafa Witten twist

$SU(2)_{twist}$:

$$Q = (4, \bar{4}) \longrightarrow (1,1)_{+-} \oplus (1,1)_{--} \oplus \cdots\cdots$$

$SO(2)_L$ chirality

$$\Longrightarrow \boxed{\text{2d } (0,2) \text{ SUSY.}}$$

② 6d (0,2) on $S^2 \times M_4$: Topological Twists

$\boxed{(0,2)\text{ Twist:}}$

$$Sp(4)_R \to \boxed{SU(2)_R} \times SO(2)_R$$

$$SO(6)_L \to \boxed{SU(2)_\ell} \times SU(2)_r \times SO(2)_L$$

"Vafa-Witten" like

→ [Assel, SSN, Wong]

$$\Rightarrow \boxed{\text{2d } (0,2) \text{ susy}}$$

$\boxed{(1,1)\text{ Twist:}}$

$$Sp(4)_R \to \boxed{SU(2)_1 \times SU(2)_2}$$

$$SO(6)_L \to \boxed{SU(2)_\ell \times SU(2)_r} \times SO(2)_L$$

SO(4) twist

"GL"-like (geometric Langlands).

$$\Rightarrow \boxed{\text{2d } (1,1) \text{ susy}}$$

[Bak, Gustavsson]
[Lawrie, SSN, Wong]

Related setup:

$$\text{M5 on } T^2 \Rightarrow \mathcal{N}=4 \text{ 4d SYM}$$

[Gadde, Gukov, Putrov]

$$\begin{tikzcd}[column sep=small]
 & \boxed{\text{M5 on } T^2 \times M_4} \arrow[dl, "\mathrm{vol}(T^2)\to 0"'] \arrow[dr] & \\
\text{VW twisted } \mathcal{N}=4 \text{ SYM} & & \text{2d } (0,2) \text{ theory on } T^2
\end{tikzcd}$$

Dictionary primarily developed for M_4 w/ boundary:

$\Rightarrow$ relates to 3d-3d duality. (similar to 3d3d from AGT).

Key Question: For $\boxed{\text{M5 on } S^2 \times M_4}$ determine 4d TFT:

$$\begin{tikzcd}[column sep=small]
 & \text{M5 on } S^2 \times M_4 \arrow[dl] \arrow[dr] & \\
\boxed{? \text{ on } M_4} & & \boxed{\text{2d } (0,2) \text{ theory on } S^2}
\end{tikzcd}$$

$\boxed{\text{(0,2) Twist}}$

To define the 2d theory on S^2 twist also $SO(2)$:

$$SO(6)_L \times Sp(4)_R \longrightarrow \boxed{SU(2)_{\text{twist}}} \times SU(2)_r \times SO(2)_{\text{twist}}$$

Scalars $\quad (1,5) \longrightarrow (1,1)_2 \oplus (1,1)_{-2} \oplus \underline{(3,1)_0}$

self dual 2-forms on M_4

Fermions $\quad (\bar{4},4) \longrightarrow (2,2)_0 \oplus (2,2)_2 \oplus (1,1)_{-2} \oplus \underline{(3,1)_{-2}} \oplus (1,1)_0 \oplus \underline{(3,1)_0}$

$B_{\mu\nu} \quad (15,1) \longrightarrow (1,1)_0 \oplus \underline{(3,1)_0} \oplus (1,3)_0 \oplus (2,2)_{+2} \oplus (2,2)_{-2}$

Dim. reduction on S^2

$\longrightarrow$ couple to sugra $\rightsquigarrow$ masses to some fields, but we expect $\underline{\text{in 4d self-dual 2-forms}}$

$\longrightarrow$ Compatible w/: M-theory realization as [Gauntlett, Kim]

$\underline{M5 \text{ on coassociative } M_4 \text{ in } G_2 \rightsquigarrow \Omega^{2+}(M_4)}$

3 $6d \to 5d$

Troublesome question: How to dim. reduce a theory, you do not know?

<u>Trick</u>: use relation to 5d SYM.

sugra (conformal) — 6d (0,2) Abelian ≡ tensor multiplet

$$\downarrow S^1$$

sugra — 5d SYM ⟶ nonabelianize

Write S^p as circle-fibration over B_{p-1}

From this point of view:

$$\boxed{\text{M5 on } S^p \times \mathbb{R}^{6-p} \triangleq \text{5d SYM coupled to sugra, on } B_{p-1}}$$

P=3 [Cordova, Jafferis] $S^1 \to S^3$ Hopf fibration.

$$\begin{array}{ccc} S^1 & \to & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

$\Rightarrow$ 6d on S^3 (w/ sugra background)

$\downarrow$ S^1 Hopf fiber

5d SYM on S^2 (w/ sugra background)

$\downarrow$ S^2

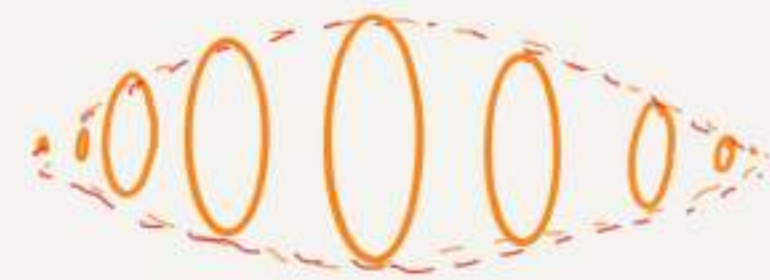
3d complex Chern Simons

Note:
- Complexification via gauge fields from $B_{\mu\nu}$
- Fermions become FP ghosts for gauge fixing.

$\Rightarrow$ derivation of 3d3d duality.

$\underline{\underline{p=2}}$ $S^2 \times \mathbb{R}^4$

$$\begin{array}{ccc} S^1 & \longrightarrow & S^2 \\ & & \downarrow \\ & & I = [0, \pi] \end{array}$$



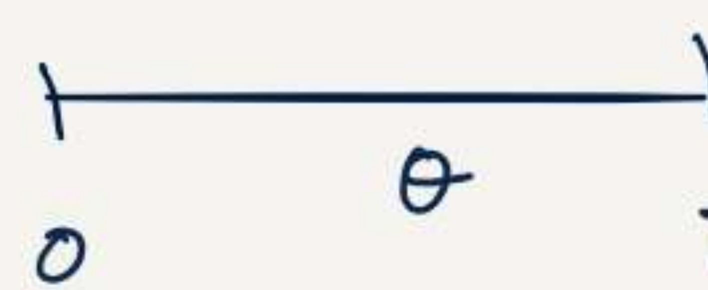
$$ds^2 = ds^2_{\mathbb{R}^4} + r^2 d\theta + \ell^2(\theta)\, d\phi^2$$

SUSY $\Rightarrow$ R-symmetry background gauge field

$\Rightarrow$ Top. twist of (0,2) theory.

$\Rightarrow$ Allow general $\ell(\theta)$ as long as @ ∂I: $\ell(\theta) \to 0$.

- $\ell(\theta) = r \sin(\theta)$ for $\underline{\text{round } S^2}$
- $\ell(\theta) = \text{const.}$ $\underline{\text{cylinder limit}}$



Requires singular b.c. (see later)

6d on S^2 abelian		5d on I abelian		5d on I non-abelian
6d Abelian EOMs	$\xrightarrow{S^1}$	**Abelian 5d SYM in sugra background**	$\longrightarrow$	**5d Action & Non-abelianization**

$$H^- = 0$$

$$D_V^2 \Phi - d\,\Phi = 0$$

$$\not{D}_V\, \rho = 0$$

V, d = sugra background (R-symmetry gauge field)

$\Rightarrow$ Preserves two <u>constant</u> Killing spinors.

$$5 \cong \Phi \;\nearrow\; \boxed{\varphi^a \quad \underline{3}} \qquad \searrow\; \boxed{\varphi = \varphi^4 + i\varphi^5}$$

$$H \to dA \;\nearrow\; A_\mu \text{ gauge field} \qquad \searrow\; \boxed{A_\theta}$$

$$\rho \longrightarrow \rho_\pm^{(i)} \qquad i = 1, 2$$

$\pm$ = 5d chirality.

SUSY & finiteness of action determines b.c. on ∂I.

BPS Eq. for 5d SYM on I : $\delta \rho = 0$

$$\boxed{D_\theta \varphi^a - \frac{1}{2}\epsilon^a{}_{bc}[\varphi^b, \varphi^c] = 0}$$

Nahm equations.

$$D_\mu \varphi^a = 0$$

$$\varphi = \bar{\varphi} = F_{\mu\nu} = F_{\mu\theta} = 0$$

Boundary Conditions : SUSY & finiteness of action

$\Rightarrow$ Requires adding extra boundary term.

$\Rightarrow$ B.c. for scalars : Nahm poles

$$\boxed{\varphi^a = \frac{\rho(\tau^a)}{\theta} + \varphi^a_0 + O(\theta)}$$

$$\rho : \; SU(2) \longrightarrow \mathfrak{g} = \text{gauge algebra}$$
$$\tau^a \longrightarrow \rho(\tau^a)$$

* $\rho = [k] : S^2 \; \mathfrak{g} = u(k)$

* ρ general: S^2 w/ punctures

[Gaiotto, Moore, Tachikawa]

4 $5d \to 4d$

Two approaches: (A) Dim reduce to 4d & then topologically twist along $\mathbb{R}^4 / M_4$

(B) Top twist 5d SYM along $\mathbb{R}^4/M_4$, then reduce.

(B) gives setup related to [Witten] 5d top twisted SYM on $\mathbb{R}^+$

Fields after twist

gauge field	A_μ
self-dual 2 form	$B_{\mu\nu} = j^a_{\mu\nu}\varphi_a$ ← 3 local SD 2 forms
Scalars	$\varphi, \bar{\varphi}$
Twisted Fermions	$\psi^{(1)}_\mu, \psi^{(2)}_\mu$ (1 form)
	$\eta^{(1)}, \eta^{(2)}$ (scalar)
	$\chi^{(1)}_{\mu\nu}, \chi^{(2)}_{\mu\nu}$ (SD 2 forms)

BPS equations = Twisted version of Nahm equations:

$$\boxed{D_\theta B_{\mu\nu} - \frac{1}{2}[B_{\mu\rho}, B_\nu{}^\rho] = 0}$$

$\Rightarrow$ 5d twisted SYM on I_θ

Bosonic: $A_\mu, A_\theta, B_{\mu\nu}, \varphi, \bar\varphi$

Fermionic: $\psi^{(i)}_\mu, \eta^{(i)}, \chi^{(i)}_{\mu\nu}$

$$
\begin{aligned}
S_F &= -\frac{r}{8\ell}\int d\theta d^4x\sqrt{|g_4|}\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}+\frac{2}{r^2}(\partial_\mu A_\theta-\partial_\theta A_\mu+[A_\mu,A_\theta])^2\right)\\
S_{\text{scalars}} &= -\frac{1}{4r\ell}\int d\theta d^4x\sqrt{|g_4|}\,\mathrm{Tr}\left(\frac{1}{4}D^\mu B_{\rho\sigma}D_\mu B^{\rho\sigma}+\frac{1}{4r^2}D_\theta B_{\rho\sigma}D_\theta B^{\rho\sigma}\right.\\
&\qquad\left.+D^\mu\varphi D_\mu\bar\varphi+\frac{1}{r^2}D_\theta\varphi D_\theta\bar\varphi\right)\\
S_\rho &= \frac{2i}{r\ell}\int d\theta d^4x\sqrt{|g_4|}\,\mathrm{Tr}\Big[\eta^{(2)}D_\mu\psi^{(1)\mu}-\psi^{(1)}_\mu D_\nu\chi^{(2)\mu\nu}+\eta^{(1)}D_\mu\psi^{(2)\mu}-\psi^{(2)}_\mu D_\nu\chi^{(1)\mu\nu}\\
&\qquad+\frac{1}{r}\left(\psi^{(1)}_\mu D_\theta\psi^{(2)\mu}-\eta^{(1)}D_\theta\eta^{(2)}-\frac{1}{4}\chi^{(1)}_{\mu\nu}D_\theta\chi^{(2)\mu\nu}\right)\Big]\\
S_{\text{Yukawa}} &= -\frac{i}{r^2\ell}\int d\theta d^4x\sqrt{|g_4|}\,\mathrm{Tr}\left(-\frac{1}{2}B_{\mu\nu}\left[\eta^{(2)},\chi^{(1)\mu\nu}\right]+\frac{1}{2}B_{\mu\nu}\left[\eta^{(1)},\chi^{(2)\mu\nu}\right]\right.\\
&\qquad-\frac{1}{2}B_{\mu\nu}\left[\chi^{(2)\mu\tau},\chi^{(1)\nu}{}_\tau\right]-2B_{\mu\nu}\left[\psi^{(2)\mu},\psi^{(1)\nu}\right]\\
&\qquad+\bar\varphi\left[\eta^{(1)},\eta^{(1)}\right]+\frac{1}{4}\bar\varphi\left[\chi^{(1)}_{\mu\nu},\chi^{(1)\mu\nu}\right]+\bar\varphi\left[\psi^{(1)}_\mu,\psi^{(1)\mu}\right]\\
&\qquad\left.-\varphi\left[\eta^{(2)},\eta^{(2)}\right]-\frac{1}{4}\varphi\left[\chi^{(2)}_{\mu\nu},\chi^{(2)\mu\nu}\right]-\varphi\left[\psi^{(2)}_\mu,\psi^{(2)\mu}\right]\right)\\
S_{\text{quartic}} &= -\frac{1}{16r^3\ell}\int d\theta d^4x\sqrt{|g_4|}\,\mathrm{Tr}\left(\frac{1}{4}\left[B_{\mu\rho},B_\nu{}^\rho\right]\left[B^\mu{}_\sigma,B^{\nu\sigma}\right]+[B_{\mu\nu},\varphi][B^{\mu\nu},\bar\varphi]-[\varphi,\bar\varphi][\varphi,\bar\varphi]\right)\\
S_{\text{bdry}} &= \frac{1}{16r^3\ell}\int d\theta d^4x\sqrt{|g_4|}\,\mathrm{Tr}\left(\partial_\theta B_{\mu\nu}[B^{\mu\rho},B^\nu{}_\rho]\right)\,.
\end{aligned}
\tag{F.11}
$$

boundary term for susy.

$\delta A_\theta = \epsilon\,\eta^{(1)}$

$\delta A_\mu = \epsilon\,\psi^{(1)}_\mu$

$\delta B_{\mu\nu} = \epsilon\,\chi^{(1)}_{\mu\nu}$

$\delta\varphi = 0$

$\delta\bar\varphi = \epsilon\,\eta^{(2)}$

$\delta\psi^{(1)}_\mu = \epsilon D_\mu\varphi$

$\delta\psi^{(2)}_\mu = \epsilon F_{\mu\theta} + \epsilon (DB)_\theta$

$\delta\eta^{(1)} = \epsilon D_\theta\varphi$

$\delta\eta^{(2)} = \epsilon[\varphi,\bar\varphi]$

$\delta\chi^{(1)}_{\mu\nu} = \epsilon[\varphi, B_{\mu\nu}]$

$\boxed{\delta\chi^{(2)}_{\mu\nu} = \epsilon F^+ + \frac{1}{r^2}\epsilon\,\text{Nahm}}$

4d Theory : * localizes around solutions of Nahm eqs.

* $r \to 0$ limit : expansion of action. (effective description at energies small compared to $1/r$).

* $\frac{1}{r}$ expansion : - gauge field kinetic term
  - $\varphi, \bar{\varphi}$ kinetic terms

  } subleading. $O(r)$.

$\Rightarrow$ $\boxed{\text{4d gauge field } A_\mu \text{ nondynamical.}}$

Reduce either twisted 5d SYM or untwisted one

$\Rightarrow$ φ^a & A_θ localize on solutions to Nahm.

$\Rightarrow$ $\boxed{\text{4d fields are valued in moduli space of Nahm eq. w/ Nahm pole b.c. at } \partial I.}$

Let $\mathcal{M}_k$ = Nahm moduli space, ie. mod. space of solutions $\{\varphi^a\}$ satisfying Nahm eqs. & b.c. w/ $\rho = [k]$:

$$\boxed{\underset{\partial_\theta + A_\theta}{D_\theta} \varphi^a - \frac{\epsilon^a{}_{bc}}{2}[\varphi^b, \varphi^c] = 0}$$

$\cong$ k-centered monopole moduli space for SU(2).

$$\Longrightarrow \boxed{\begin{aligned} &\varphi^a \overbrace{(\theta, x^\mu)}^{I \times \mathbb{R}^4} = \varphi^a(\theta, X(x^\mu)) \\ &\delta\varphi^a = \Upsilon^{\hat{a}}_I \, \delta X^I \end{aligned}}$$

X^I = coordinates on $\mathcal{M}_k$

$\Upsilon^{\hat{a}}_I$ = basis of $T^*\mathcal{M}_k$.

$\Longrightarrow$ Moduli space expansion of φ^a (& A_θ).

<u>Fermions:</u> satisfy susy var. of Nahm: " $D_\theta \rho + [\varphi^a, \rho] = 0$ ".

4d Reduction: (on $\mathbb{R}^4$)

$\boxed{\text{4d } N=2 \ \sigma\text{-model}}$ into $\boxed{\mathcal{M}_k = k\text{-monopole moduli space.}}$

$X^I : M_4 \to \mathcal{M}_k$

$\xi^{(1,2)} \in \Gamma(X^* T\mathcal{M}_k \otimes S^I)$ $\qquad S^I = $ spin bdle on M_4

$$\boxed{\begin{aligned} S_{4d} = \frac{1}{4r}\int d^4x \sqrt{|g_4|}\Big(G_{IJ}\big(\partial_\mu X^I \partial^\mu X^J - 2i\,\xi^{(1)I}\sigma^\mu D_\mu \xi^J\big) \\ - \frac{1}{2} R_{IJKL}\big(\xi^{(1)I}\xi^{(1)J}\big)\big(\xi^{(2)K}\xi^{(2)L}\big)\Big)\end{aligned}}$$

G_{IJ} metric on $\mathcal{M}_k$

R Riemann on $\mathcal{M}_k$

$\mathcal{M}_k = $ Hyper Kähler $\Rightarrow$ $\overset{SO(4k)}{I}$ index decomposes into $\overset{Sp(k)}{2k} \otimes \overset{Sp(1)}{2}$

$\Rightarrow$ S_{4d} is equiv. to Bagger Witten model with target $= \mathcal{M}_k$.

(5) $\mathbb{R}^4 \to M_4$: Topological Twist along $\mathbb{R}^4$

Note: by following (B) we can apply similar analysis & arrive at a top. σ-model into the mod. space of Nahm (twisted) eqs.

Key problem in both approaches:

X^I transform under $SU(2)_\ell \subseteq SO(4)_L$ of M_4

cf. similar situation for $N=4$ SYM on $\Sigma \times C$ [Bershadsky, Johanssen, Sadov, Vafa]

$\Rightarrow$ Requires knowledge of geometry of $\mathcal{M}_K$.

$\Rightarrow$ $K = 1, 2$ ✓ ($\to$ see later)

A simpler case, where this problem does not occur:

$$\boxed{M_4 = \text{HyperKähler (HK)}}$$

($\underline{SU(2)_\ell \text{ connection}}$ is $\underline{\text{trivial}}$)

$M_4 = HK$ (ie. K3 or T^4) : $SO(4)_L \longrightarrow SU(2)_\ell$ on M_4

$$\left.\begin{aligned} X^I &\longrightarrow X^I \\ \xi^{(1)} &\longrightarrow \lambda^I,\ \chi_{\mu\nu}{}^I \\ \xi^{(2)} &\longrightarrow \gamma_\mu{}^I \end{aligned}\right\}$$

twisting has no effect on I index

$\Rightarrow$ Top. σ-model localizes on $\boxed{\text{Tri-holomorphic maps}}$

X^I: $M_4 \longrightarrow \mathcal{M}_K$ = K-monopole mod. space.

j^a ω^a HK structures

$a = 1, 2, 3$

$$\boxed{\partial_\mu X^I - j^{a}_{\mu}{}^{\nu} \partial_\nu X^J \omega^{a}_{J}{}^{I} = 0}$$

This model appeared in Anselmi - Fré for $M_4 \to \mathcal{M}$ = some HK target.

Topological invariant action:

$$S_{M_4=HK} = QV - \underbrace{\frac{1}{2\pi e}\int_{M_4} j^a \wedge X^* \omega^a}_{\text{topological invariant of } M_4}$$

And EM tensor $T = QW$.

$\Rightarrow \quad M_4 = HK \quad \Rightarrow$ 4d topological σ-model into the k-monopole moduli space $\mathcal{M}_k$

$\{X_I, \chi_\mu^I, \lambda^I, \chi^I_{\mu\nu}\}$

General M_4 : $SO(4)_L \cong SU(2)_l \times SU(2)_r$

X^I transform under $SU(2)_l$

$\Rightarrow$ Some components become S.D. 2-forms.

$\Rightarrow$ 4d Top. Theory w/ scalars, S.D. 2-forms.

In the literature: [Kapustin, Vyas] (different action).

$\Rightarrow$ cf. [Bershadsky, Johansen, Sadov, Vafa]: $N=4$ w/ VW twist on $C \times \Sigma$: σ-model $X^I : \Sigma \to \mathcal{M}_C^{\text{Hitchin}}$

Twist along Σ: X^I transform under $U(1)_\Sigma$.

$$\begin{array}{ccl} \mathbb{C}^{3g-3} \longrightarrow & \mathcal{M}_C^{\text{Hitchin}} & \text{w/ } U(1)_\Sigma \text{ acts on fibers} \\ & \downarrow & \text{by rotation.} \\ & B & \end{array}$$

$\Rightarrow$ Require detailed information on $\mathcal{M}_C$ in our case.

Nahm equations, poles & Monopoles

Nahm, Hitchin, Donaldson...

$\boxed{\text{Solution to Nahm's eq. on } I \text{ w/ Nahm pole b.c.}}$ $\xleftrightarrow{\text{Hitchin}}$ $\boxed{k\text{-centered } SU(2) \text{ monopoles}}$

$$F_A = - * F_A \qquad A = d\theta\, A_\theta + \sum_{a=1}^{3} \varphi^a dx_a$$

Geometric Properties:

reduced mod space

$\boxed{\text{Moduli space } \mathcal{M}_k \text{ Hyper-Kähler}}$ $\longleftarrow$ $\boxed{\mathcal{M}_k = \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_k^0}{\mathbb{Z}_k}}$

center of mass.

$\updownarrow$ Bielawski

$\boxed{\mathcal{M}_k^0 \text{ description as Slodowy slices in } T^*SU(k)_{\mathbb{C}}.}$

$k=1$: $\mathbb{R}^3 \times S^1$

$k=2$: $\mathcal{M}_2^0 = \mathcal{M}_{\text{Atiyah Hitchin}}.$

$\boxed{K=1}$ $\boxed{\mathcal{M}_1 = \mathbb{R}^3 \times S^1_\phi}$, M_4 general, $X^I : M_4 \to \mathcal{M}_1$

$$X^I \begin{cases} \to X^\phi \\ \to \boxed{X^a j_{a\mu\nu} = B_{\mu\nu}} \quad \text{S.D. 2-form.} \end{cases}$$

$$\xi^{(1),(2)} \begin{cases} \to \eta, \ \chi_{\mu\nu} \\ \to \psi_\mu \end{cases} \Big\} \text{ twisted fermions}$$

single twisted hypermultiplet.

$$\Rightarrow \boxed{S_{k=1} = \frac{1}{4r}\int_{M_4} \left(* d\phi \wedge d\phi + * dB \wedge dB + 2\psi \wedge (* d\eta - d\chi) \right)}$$

Top SUSY:

$$\delta\phi = \epsilon\eta$$

$$\delta B_{\mu\nu} = \epsilon\chi_{\mu\nu}$$

$$\delta\eta = \delta\chi = 0$$

$$\delta\psi_\mu = \epsilon \underbrace{(\partial_\mu\phi + \partial^\nu B_{\nu\mu})}_{=0} \Rightarrow$$

Localizes on constant ϕ & $B \in H^{2+}(M_4)$

Note: similarity Kapustin-Vyas, 4d top. A-model.

$\boxed{K=2}$: $\boxed{\mathcal{M}_2 = \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_{AH}}{\mathbb{Z}_2}}$ (Atiyah-Hitchin manifold).

$\mathcal{M}_{AH}$ has explicitly known Hyper-Kähler metric:

$$\longrightarrow \quad ds^2_{AH} = f(r)^2 dr^2 + a(r)^2 \underline{\sigma_1}^2 + b(r)^2 \underline{\sigma_2}^2 + c(r)^2 \underline{\sigma_3}^2$$

- $\sigma_i = SO(3)$ invariant 1-forms.
- $\frac{da}{dr} = \frac{f}{2bc}(b^2 + c^2 - a^2 - 2bc)$

$\longrightarrow$ $SU(2)_\ell$ is topologically twisted w/

$$\mathrm{diag}\left(\underset{\substack{\uparrow \\ \text{HK rotations on AH manifold}}}{SO(3)_{AH}} \times \underset{\substack{\uparrow \\ /\ \mathbb{R}^3}}{SO(3)_{abelian}}\right) \Longrightarrow X^I \begin{array}{l} \nearrow \text{scalars } \phi_i \\ \searrow \text{SD 2 forms } \beta_i \end{array}$$

$$\Longrightarrow S_{k=2} = \frac{1}{4\pi}\int_{M_4} d\beta^i \wedge * d\beta_i + d\phi^i \wedge * d\phi_i + \text{fermions.}$$

Remarks: Clearly: this is just the 1st step.

$k = 1$ & 2: explicit construction of the 4d top. σ-model.

* $S = QV$ (Q-exact action & $S_T = 0$).
* Theory of scalars & S.D. 2-forms
  $\Rightarrow$ What are BPS observables?
* Some similarity to KV A-model.
  $\Rightarrow$ What is 3d $N=4$ dim. reduction here?

$k \geq 3$: * Alternative way to characterize $SU(2)_\ell$ transformation?
  $\rightsquigarrow$ Connection to Bielawski description of $\mathcal{M}_k$
  $\rightsquigarrow$ twistor? Sim. Fibration structure as $\mathcal{M}^{\text{twistor}}$?

6 Conclusions

Executive summary:

6d (0,2) on $S^2 \times M_4$ (U(k))

4d topological σ-model
$M_4 \longrightarrow \mathcal{M}_K$
w/ scalar & SD 2-forms

2d (0,2) theory
$T[M_4]$

Clearly next step is setting up the precise dictionary for this 4d-2d correspondence.

$\Longrightarrow$ data on both sides ($\rightsquigarrow$ Closset's talk for (0,2) loc.).

Remarks

1) Despite lack of 6d description of (0,2) theory on N M5, we can use the relation to 5d SYM (+ sugra) to determine dim. Reductions on <u>S^1-fibrations</u>.

$$
\begin{array}{lccccl}
 & S^1 & & & & \\
 & \downarrow & & & & \\
\text{[Assel, SSN, Wang '16]} & S^2 & \longrightarrow & I & & \text{4d Topol. } \sigma\text{-model } \mathcal{M}_K \to \text{4d2d} \\
\text{[Cordova, Jafferis '13]} & S^3 & \longrightarrow & S^2 & & \text{3d cx Chern Simons} \rightsquigarrow \text{3d3d} \\
\text{[Cordova, Jafferis '16]} & S^4 & \longrightarrow & (S^2 \to I) & & \text{2d Toda Theory.} \rightsquigarrow \text{AGT} \\
 & \text{6d} & & \text{5d + sugra} & &
\end{array}
$$

Application to other reductions:

* elliptic fibrations [Assel, SSN, to appear].
* Topological twisted versions of AGT [Gukov, Putrov, Vafa '16]

2) 4d/2d correspondence

* Explore relation to $T^2 \times M_4$ w/ VW twist on M_4 [Ghade, Gukov, Putrov].

  $T^2 \times M_4 \to$ (0,2) elliptic genus.

  $T^2 \times M_4 \to$ VW partition function.

* Setup dictionary: $S^2 \times M_4$

  S^2 ← half-twisted (0,2)

  M_4 ← 4d σ-model

* Observables of 4d theory (Q-cohomology including SD 2-form field)

* 2d (0,2) w/ topolog. twist: localization computations. Carry from (2,2) locus).

3) Relation to 3d $N=4$ Theories & Mirror symmetry

The 4d topological theory is related to 3d $N=4$ top. twisted A-model (hyper):

4d σ-model

$N=2$ $\boxed{4d}$ A gauge th. (Donaldson Witten) $\qquad$ $\boxed{\text{A hyper multiplet}}$

$\downarrow S^1$ $\qquad\qquad$ $\downarrow S^1$

$N=4$ $\boxed{3d}$ $\boxed{A}$ twist $\qquad$ $\boxed{A}$-hyper (Kapustin Vyas)

$\boxed{B}$ twist $\qquad$ $\boxed{B}$-hyper (Rozansky Witten)

Example: "particle vortex duality"

$U(1)$ twisted DW $\longleftrightarrow$ Hyper multiplet w/ target $\mathbb{R}^3 \times S^1$

$\boxed{A}$ (from dim. red. of DW) $\qquad$ $\boxed{A}$ KV w/ target S^1

$\boxed{B}$ ($A + i\phi$ complex gauge field) $\qquad$ $\boxed{B}$ RW w/ target $T^*\mathbb{C}^*$