New supersymmetric index of heterotic compactifications with flux

D. Israël and M. Sarkis (1509.05704)

Introduction

- Little is known concerning torsional heterotic compactifications
- $\mathcal{N} = 2$ compactifications are crucial in the context of dualities
- Heterotic string: allows for worldsheet description of fluxes
- Microscopic GLSM description for them
  - Melnikov, Quigley, Sethi, Stern (hep-th/1212.1212), Adams, Ernebjerg and Lapan (hep-th/0611084): Torsion GLSM
  - Now: Computation of the Fu-Yau non-holomorphic genus
  - Work in progress with D.Israël and C.Angelantonj: computation of the gauge and gravitational threshold corrections

Content

- $\mathcal{N} = 1$ and $\mathcal{N} = 2$ heterotic compactifications
- Fu-Yau Geometry and Torsion gauged linear sigma model
- Fu-Yau non-holomorphic genus and Localization
- Geometrical formula

Supersymmetry constraints

Strominger's system gives the conditions for $\mathcal{N} = 1$ supersymmetry in spacetime:

- $d\left(||\Omega||_J J \wedge J\right) = 0$
- $F^{(2,0)} = F^{(0,2)} = 0$
- $J^{i\bar{\jmath}} F_{i\bar{\jmath}} = 0$
- $\mathrm{d}H = 2i\partial\bar{\partial}J = \frac{\alpha'}{4}\left(\mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F)\right)$

Examples of solutions:

- Calabi-Yau with standard embedding
- Calabi-Yau equipped with a more general holomorphic vector bundle
- $T^2 \hookrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{S}$ with $\mathcal{S}$ a $K3$ surface: example of non-Kähler geometry known as Fu-Yau geometry; Non-vanishing H-flux

Fu-Yau geometry

- Golstein and Prokushkin: $SU(3)$ structure made explicit
- Metric: $\boxed{ds^2 = \frac{U_2}{T_2}|\iota|^2 + e^{2\Delta(y)} ds^2(\mathcal{S})}$, $\iota = dx^1 + T dx^2 + \pi^\star \alpha$
- Torsion: $H = \star_{\mathcal{S}} \mathrm{d} e^{2\Delta} - \frac{U_2}{T_2} \mathrm{Re}\left(\bar{\iota} \wedge \star_{\mathcal{S}} \mathrm{d}\iota\right)$
- Define the complex two-form $\omega = \omega_1 + T\omega_2$ by $\frac{1}{2\pi} d\iota = \pi^\star \omega$
- Need to impose:
  - $\omega_i \in H^2(\mathcal{S}, \mathbb{Z})$
  - ω has no $\bigwedge^{0,2} T^* \mathcal{S}$ component
  - Primitivity with respect to the base, i.e. $\omega \wedge J_{\mathcal{S}} = 0$
- Imposing further $\omega \in H^{1,1}(\mathcal{S})$ leads to extended $\mathcal{N} = 2$ susy in spacetime
- Vector bundle over $\mathcal{M}$: pullback of a stable holomorphic vector bundle over $\mathcal{S}$

$(0,2)$ superspace

- $(\sigma^+, \sigma^-, \theta, \bar{\theta})$, $D_+ = \partial_\theta - i\bar{\theta}\nabla_+$ and $\bar{D}_+ = -\partial_{\bar{\theta}} + i\theta\nabla_+$
- Vector superfields in Wess-Zumino gauge:

$$\mathcal{V} = A_- - 2i\theta\bar{\mu} - 2i\bar{\theta}\mu + 2\theta\bar{\theta}D\,, \qquad \mathcal{A} = \theta\bar{\theta}A_+\,.$$

- Chiral superfields ($\bar{D}_+\Phi = 0$): $\Phi = \phi + \sqrt{2}\theta\psi - i\theta\bar{\theta}\nabla_+\phi$.
- Fermi superfields ($\bar{D}_+\Gamma = 0$): $\Gamma = \gamma + \sqrt{2}\theta G - i\theta\bar{\theta}\nabla_+\gamma$.
- Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\int \mathrm{d}^2\theta\,\bar{\Phi}(\partial_- + iQ\mathcal{V})\Phi - \frac{1}{2}\int \mathrm{d}^2\theta\,\bar{\Gamma}\Gamma - \frac{1}{8e^2}\int \mathrm{d}^2\theta\,\bar{\Upsilon}\Upsilon$$
$$- \int \mathrm{d}\theta\,\Gamma J(\Phi) + \frac{h}{4}\int \mathrm{d}\theta\,\Upsilon + \mathrm{h.c.}\,.$$

- $J(\Phi)$ holomorphic homogeneous polynomial.
- $\Upsilon = \bar{D}_+(\partial_-\mathcal{A} + i\mathcal{V})$ superfield strengh (chiral superfield)

GLSM: $K3$ base

- $K3$ base: standard GLSM with superpotential
  $\mathcal{L}_{\mathrm{J}} = \int d\theta^+ \left(\tilde{\Gamma}_\alpha G^\alpha(\Phi_i) + P\,\Gamma_a\, J^a(\Phi_i)\right) + h.c.$
- Complete intersection $\bigcap_{\alpha=1}^{p} \{\phi_i \mid G^\alpha(\phi_i) = 0\}$
- Holomorphic vector bundle
  $0 \longrightarrow \mathcal{V} \xrightarrow{\iota} \bigoplus_{a=0}^{s} \mathcal{O}(Q_a) \xrightarrow{\otimes J^a} \mathcal{O}(-Q_P) \longrightarrow 0$
- Anomaly $\propto$ $\boxed{\left(\sum_{\text{Chiral}} Q^2 - \sum_{\text{Fermi}} Q^2\right) \int \mathrm{d}\theta\, \Xi \Upsilon}$
- $\sum_{\text{Chiral}} Q^2 - \sum_{\text{Fermi}} Q^2 \propto \mathrm{ch}_2(\mathcal{V}) - \mathrm{ch}_2(\mathcal{T}_{\mathcal{S}})$
- The theory is at this point ill-defined quantum mechanically

GLSM: Torus fibration

- For a generic non-ortogonal torus with B-field:

$$g := \frac{U_2}{T_2}\begin{pmatrix} 1 & T_1 \\ T_1 & |T|^2 \end{pmatrix}, \; b := \begin{pmatrix} 0 & U_1 \\ -U_1 & 0 \end{pmatrix}, \; E := g + b$$

- "Shift multiplets" $\Omega_{1,2} = (\omega_{1,2}, \chi_{1,2})$ chiral with axial coupling to the gauge field

$$\boxed{\mathcal{L} = -\frac{iE_{ij}}{8}\int \mathrm{d}^2\theta \left(\Omega_i + \bar{\Omega}_i + 2M_i\mathcal{A}\right)\left(\partial_-(\Omega_j - \bar{\Omega}_j) + 2iM_j\mathcal{V}\right) - \frac{ih_i}{4}\int \mathrm{d}\theta\, \Omega_i \Upsilon \; + h.c.}$$

- Decoupling of the real part of Ω_i while preserving supersymmetry $\to$ vanishing of the couplings to the gaugino:

$$\frac{U_2}{T_2}(M_1 + T_1 M_2) - U_1 M_2 + h_1 = 0$$

$$\frac{U_2}{T_2}\left[(M_1 + T_1 M_2)T_1 + T_2^2 M_2\right] + U_1 M_1 + h_2 = 0\,.$$

- Torsion multiplet

$$\Theta := \Big(\alpha := \mathrm{Im}(\omega_1) + T\mathrm{Im}(\omega_2), \quad \chi := \mathrm{Re}(\chi_1) - T\mathrm{Re}(\chi_2)\Big),$$

$$M := M_1 + TM_2$$

Anomaly cancellation condition and quantization conditions

- This Lagrangian is NOT gauge invariant
- It satisfies $\delta_{\Xi}\mathcal{L} = \frac{U_2}{2T_2}|M|^2 \int d\theta^+ \,\Upsilon\Xi + h.c.$
- Combines with the base anomaly to give the consistency condition

$$\boxed{\sum_{\text{Chiral}} Q^2 - \sum_{\text{Fermi}} Q^2 - \frac{2U_2}{T_2}|M|^2 = 0}$$

- Corresponds to the tadpole condition obtained by integrating the Bianchi identity

$$\boxed{\mathrm{ch}_2(\mathcal{V}) - \mathrm{ch}_2(\mathcal{T}_{\mathcal{S}}) - \frac{U_2}{T_2}\omega \wedge \star_{\mathcal{S}}\bar{\omega} = 0}$$

- h_1 and h_2 quantized $\rightarrow$ quantization of torus moduli: $U, T \in \mathbb{Q}(\sqrt{D})$ with $D = b^2 - 4ac$ the discriminant of some positive integer quadratic form $\Leftrightarrow$ rationality of the T^2-valued CFT

Fu-Yau non-holomorphic genus

- Elliptic genus $Z_{\rm ELL} = \mathrm{Tr}_{\rm RR}\left((-1)^F e^{2i\pi z J_0} q^{\Delta}\bar{q}^{\bar{\Delta}}\right)$ vanishes here because of χ
- Fu-Yau non-hol genus $\boxed{Z_{\rm FY} = \frac{1}{\bar{\eta}^2}\mathrm{Tr}_{\rm RR}\left((-1)^F e^{2i\pi z J_0}\bar{J}_0 q^{\Delta}\bar{q}^{\bar{\Delta}}\right)}$
- Coupling to a background left-moving $U(1)$ gauge field to impose twisted boundary conditions, $a_{\rm L} = \frac{\pi\bar{z}}{2i\tau_2}\mathrm{d}w - \frac{\pi z}{2i\tau_2}\mathrm{d}\bar{w}$
- $\bar{J}_0 = \bar{\chi}\chi +$ (do not saturate the fermionic measure)

$$Z_{\rm FY}(\tau,\bar{\tau},z) = \frac{1}{\bar{\eta}(\bar{\tau})^2}\int \mathscr{D}a_z\mathscr{D}a_{\bar{z}}\mathscr{D}\lambda\mathscr{D}\bar{\lambda}\mathscr{D}D\, e^{-\frac{1}{e^2}S_{\rm v.m.}[a,\lambda,D]-t\,S_{\rm FI}(a,D)} \times$$
$$\times \int \mathscr{D}\phi_i\mathscr{D}\bar{\phi}_i\mathscr{D}\psi_i\mathscr{D}\bar{\psi}_i\, e^{-\frac{1}{g^2}S_{\rm c.m.}[\phi_i,\psi_i,a,D,a_{\rm L}]} \times$$
$$\times \int \mathscr{D}\gamma_a\mathscr{D}\bar{\gamma}_a\mathscr{D}G_a\mathscr{D}\bar{G}_a\, e^{-\frac{1}{f^2}S_{\rm f.m.}[\gamma_a,G_a,a,a_{\rm L}]-S_{\rm J}[\gamma_a,G_a,\phi_i,\psi_i]} \times$$
$$\times \int \mathscr{D}\alpha\mathscr{D}\bar{\alpha}\mathscr{D}\chi\mathscr{D}\bar{\chi}\, e^{-S_{\rm t.m.}[\alpha,\chi,a,a_{\rm L}]}\int\frac{\mathrm{d}^2 w}{2\tau_2}\,\bar{\chi}\chi\,,$$

Localization

- Define the supercharge $\mathcal{Q} = \left(\epsilon Q_+ - \bar{\epsilon}\bar{Q}_+ + \delta_{\text{gauge}}\right)\Big|_{\epsilon=\bar{\epsilon}=1}$ with δ_{gauge} needed to restore Wess-Zumino gauge
- Bad
  - Lagrangian not gauge invariant
  - Functional integral measure not gauge invariant
  - $\int \frac{\mathrm{d}^2 w}{2\tau_2}\, \bar{\chi}\chi$ is not supersymmetric
- Good
  - $\mathcal{L}_{\text{vm,FI,cm,fm,J}}$ are all $\mathcal{Q}$-exact
  - $\mathcal{L}_{\text{t.m.}}$ not $\mathcal{Q}$-exact but quadratic
  - $\mathcal{Q}\left(\mathscr{D}\Phi\mathscr{D}\Gamma e^{-S_{\text{t.m.}}[\Theta,A]}\right) = 0$
  - $\mathcal{Q}\left(\bar{\chi}\chi\right)$ do not saturate $\mathscr{D}\bar{\chi}\mathscr{D}\chi$
- Hence, using Stokes theorem in field space,

$$\boxed{\frac{\partial Z_{\text{FY}}(\tau,\bar{\tau},z)}{\partial(1/e^2)} = 0}$$

Localization

- Rescale the superfields: $\tilde{\Phi} \equiv \Phi/g$ and $\tilde{\Gamma} \equiv \Gamma/f$
- Localize the functional integral to the free limit $e, g, f \to 0$
- Localization locus $= \left\{u, \bar{u}, \lambda_0, \bar{\lambda}_0, D_0\right\}$
  $\left(\text{with } a = \frac{\pi \bar{u}}{2i\tau_2}\mathrm{d}w - \frac{\pi u}{2i\tau_2}\mathrm{d}\bar{w}\right)$
- The torsion multiplet contribution, though non-exact, is quadratic, hence can be evaluated explicitely.

K3 base contribution

- Zeta-regularized Laplacian
  $\mathrm{Det}_\zeta \nabla(u)^\dagger \nabla(u) = e^{\frac{\pi}{\tau_2}(u-\bar{u})^2} |\vartheta_1(\tau|u)|^2$
- Define $\mathrm{Det}\, \nabla(u) = e^{\frac{\pi}{\tau_2}(u^2-u\bar{u})} \vartheta_1(\tau|u)$
- Holomorphic in the sense that it is annihilated by $\frac{D}{D\bar{u}} = \frac{\partial}{\partial \bar{u}} + \frac{\pi}{\tau_2} u$: holomorphic section of a holomorphic line bundle over the space of gauge connections
- Choice consistent with torsion multiplet contribution
- Chiral multiplet $\boxed{ie^{-\frac{\pi}{\tau_2}(v^2-v\bar{v})} \frac{\eta(\tau)}{\vartheta_1(\tau|v)}}$ with $v = Q_i u + q_i^L z$
- Fermi multiplet $\boxed{ie^{\frac{\pi}{\tau_2}(v^2-v\bar{v})} \frac{\vartheta_1(\tau|v)}{\eta(\tau)}}$ with $v = Q_a u + q_a^L z$
- Vector multiplet $-2i\pi\eta(\tau)^2 du$

Torsion multiplet contribution

- For each S^1:

$$\int \mathscr{D}\varphi \;\exp\left\{-\frac{R^2}{2\pi}\int d^2w\,\left(\partial\varphi\bar{\partial}\varphi + 2a_{\bar{w}}\partial\varphi + a_w a_{\bar{w}}\right)\right\}$$

- At free fermion radius $R_{\mathrm{f}} = 1/\sqrt{2}$: infinite sum over $\widehat{u(1)}_L \times \widehat{u(1)}_R$ representations $\xrightarrow{\text{Poisson}}$ finite sum over spin structures:

$$\frac{1}{2}\frac{1}{|\eta(\tau)|^2} e^{-\frac{2\pi}{\tau_2}R_{\mathrm{f}}^2\left(v\bar{v}-v^2\right)} \sum_{k,l=0}^{1} \vartheta\begin{bmatrix}k\\l\end{bmatrix}(\tau|v)\bar{\vartheta}\begin{bmatrix}k\\l\end{bmatrix}(\bar{\tau}|0)$$

- $v = Mu + m_{\mathrm{L}}z$

Torsion multiplet contribution

- Generalization to arbitrary torus described by a generic $c = 2$ RCFT
- Hosono, Lian, Oguiso and Yau (hep-th/0211230): rational Narain lattice $\Gamma(T, U) \leftrightarrow (\Gamma_{\mathrm{L}}, \Gamma_{\mathrm{R}}, \varphi)$

$$\Gamma_{\mathrm{L}} := \Gamma(T, U) \cap \mathbb{R}^{2,0}, \qquad \Gamma_{\mathrm{R}} := \Gamma(T, U) \cap \mathbb{R}^{0,2}$$

- Corresponding partition function:

$$e^{-\frac{2\pi}{\tau_2}\left(v\bar{v}-v^2\right)||p_{\mathrm{M}}||^2} \sum_{\mu \in \Gamma_{\mathrm{L}}^{\vee}/\Gamma_{\mathrm{L}}} \frac{\Theta_{\mu}^{\Gamma_{\mathrm{L}}}(\tau|vp_{\mathrm{M}})}{\eta(\tau)^2} \frac{\bar{\Theta}_{\varphi(\mu)}^{\Gamma_R}(\bar{\tau}|0)}{\bar{\eta}(\bar{\tau})^2}$$

with $v = u + \lambda z$, $p_{\mathrm{M}} \in \Gamma_{\mathrm{L}}$

- $\Theta_{\mu}^{\Gamma}(\tau|z)$ theta function associated to the lattice Γ

Full one-loop deteminant

$$\Sigma_{\text{1-loop}} = -2i\pi\eta(\tau)^2 \prod_{\Phi_i} \frac{i\eta(\tau)}{\vartheta_1(\tau|Q_i u + q_i^{\mathrm{L}} z)} \prod_{\Gamma_a} \frac{i\vartheta_1(\tau|Q_a u + q_a^{\mathrm{L}} z)}{\eta(\tau)} \times$$

$$\times \sum_{\mu \in \Gamma_{\mathrm{L}}^{\vee}/\Gamma_{\mathrm{L}}} \frac{\Theta_{\mu}^{\Gamma_{\mathrm{L}}}(\tau|up_{\mathrm{M}})}{\eta(\tau)^2} \frac{\bar{\Theta}_{\varphi(\mu)}^{\Gamma_R}(\bar{\tau}|0)}{\bar{\eta}(\bar{\tau})^2} \times$$

$$\times \exp\left[\underbrace{\left(-\sum_i Q_i{}^2 + \sum_a Q_a{}^2 + \frac{2U_2}{T_2}|M|^2 \right)}_{=0} \left(u^2 - u\bar{u} \right) \right] du$$

with

$$p_{\mathrm{M}} = \sqrt{\frac{2U_2}{T_2}} \begin{pmatrix} M_1 + T_1 M_2 \\ T_2 M_2 \end{pmatrix} \in \Gamma_{\mathrm{L}}$$

Final formula (rank one)

$$Z_{\mathsf{FY}}(\tau,\bar\tau,z) = \pm\eta(\tau)^2 \sum_{u^\star\in\mathcal{M}^{\pm}_{\mathrm{sing}}} \oint_{\mathcal{C}(u^\star)} du$$

$$\prod_{\Phi_i}\left(i\frac{\eta(\tau)}{\vartheta_1(\tau|Q_i u + q_i^{\mathsf{L}} z)}\right) \prod_{\Gamma_a}\left(i\frac{\vartheta_1(\tau|Q_a u + q_a^{\mathsf{L}} z)}{\eta(\tau)}\right) \times$$

$$\times \sum_{\mu\in\Gamma_{\mathsf{L}}^\vee/\Gamma_{\mathsf{L}}} \frac{\Theta_\mu^{\Gamma_{\mathsf{L}}}(\tau|u p_{\mathsf{M}})}{\eta(\tau)^2} \frac{\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}$$

Geometrical formula

- Modified holomorphic Euler characteristic:

$$\boxed{Z_{\mathsf{FY}}(\mathcal{M},\mathcal{V},\omega|\tau,\bar\tau,z) = q^{\frac{r-d}{12}} y^{-\frac{r}{2}} \int_{\mathcal{S}} \mathrm{ch}\,(\mathbb{E}^{\mathcal{S},\mathcal{V}}_{q,w})\,\mathrm{td}\,(\mathcal{T}_{\mathcal{S}})\mathcal{Z}_{\mathrm{tor}}(\tau,\bar\tau,p_\omega)}$$

$$\mathcal{Z}_{\mathrm{tor}}(\tau,\bar\tau,p_\omega) = \sum_{\mu\in\Gamma_{\mathrm{L}}^\vee/\Gamma_{\mathrm{L}}} \frac{\Theta^{\Gamma_{\mathrm{L}}}_{\mu}\left(\tau\,\middle|\,\frac{p_\omega}{2i\pi}\right)}{\eta(\tau)^2}\,\frac{\bar\Theta^{\Gamma_{\mathrm{R}}}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}$$

and $p_\omega = \sqrt{\frac{2U_2}{T_2}}\begin{pmatrix}\omega_1 + T_1\omega_2\\ T_2\omega_2\end{pmatrix} \in \Gamma_{\mathrm{L}} \otimes \mathrm{Pic}(\mathcal{S})$

$([\omega_\ell] \in \mathrm{H}^2(\mathcal{S},\mathbb{Z}) \cap \mathrm{H}^{1,1}_{\bar\partial}(\mathcal{S}))$

Relation to Jacobi forms

- Z_{FY}, though non-holomorphic in τ, transforms as a weak Jacobi form of weight zero and index $\frac{r}{2}$ whenever the Bianchi identity

$$\mathrm{ch}_2(\mathcal{V}) - \mathrm{ch}_2(\mathcal{T}_{\mathcal{S}}) = \frac{U_2}{T_2}\omega \wedge \star_{\mathcal{S}} \bar{\omega}$$

  is satisfied
- In the case of a fibration $S^1 \to X_5 \to \mathcal{S}$ it corresponds to a skew holomorphic modular form defined by Skoruppa

Wilson lines

- One can consider abelian connections along the torus fibre, of the form $A = \mathcal{T}^i \,\mathrm{Re}(\bar{V}^i \,\iota)$ with $\{\mathcal{T}^i\}$ basis of the Cartan algebra of $\mathfrak{e}_8$.
- To write supersymmetric coupling in $(0,2)$ GLSM, need to first bosonize femions, complete into neutral shift multiplets:

$$\psi_- \to B = b + \sqrt{2}\theta\xi_+ - i\theta\bar{\theta}\partial_+ b$$

$$\begin{aligned}\mathcal{L}_{\text{Wilson}} = &- \frac{iR_{\mathrm{f}}^2}{8}\int \mathrm{d}^2\theta \left(B_n + \bar{B}_n\right)\partial_- \left(B_n - \bar{B}_n\right) \\ &- \frac{iq_n^l}{16}\int \mathrm{d}^2\theta \left(\Omega_l + \bar{\Omega}_l + 2m_l\mathcal{A}\right)\partial_- \left(B_n - \bar{B}_n\right)\end{aligned}$$

Wilson lines

- Leads to couplings $q_n^I \bar{\psi}_n \left(\partial_+ \left(\omega_I - \bar{\omega}_I\right) + 2 m_I A_+\right) \psi_n$
- Extra degrees of freedom:
  - $\mathrm{Re}(b_n) \to$ kill couplings to gaugino
  - $\xi_+ \to$ extra $\bar{J}_0$ insertion
  - $\partial_+ \mathrm{Im}(b_n) \to$ factorized sector by sector
- In term of a heterotic Narain lattice $\Gamma_{10,2}$

$$\sum_{\mu \in \Gamma_{\mathrm{L}}^{\vee}/\Gamma_{\mathrm{L}}} \frac{\Theta_{\mu}^{\Gamma_{\mathrm{L}}}(\tau|p_\omega)}{\eta(\tau)^2} \frac{\bar{\Theta}_{\varphi(\mu)}^{\Gamma_R}(\bar{\tau}|0)}{\bar{\eta}(\bar{\tau})^2} \to \sum_{(p_{\mathrm{L}}, p_{\mathrm{R}}) \in \Gamma_{10,2}} \frac{q^{\frac{1}{4}|p_{\mathrm{L}}|^2}}{\eta(\tau)^{18}} \frac{\bar{q}^{\frac{1}{4}|p_{\mathrm{R}}|^2}}{\bar{\eta}(\bar{\tau})^2} e^{-2i\pi \, \mathrm{Re}\left(\omega \, \overline{p_{\mathrm{L}}}|_{V^i=0}\right)}$$

Simplification

- Using a result from Gritsenko (math/9906191):

$$\theta_1(\tau|z+\xi) = \exp\left\{-\frac{\pi^2}{6}E_2(\tau)\xi^2 + \frac{\theta_1'(\tau|z)}{\theta_1(\tau|z)}\xi - \sum_{n\geqslant 2}\wp^{(n-2)}(\tau,z)\frac{\xi^n}{n!}\right\}\theta_1(\tau|z)$$

- One can show that:

$$Z_{\mathrm{FY}}(\tau,\bar\tau,z) = -(-i)^r \sum_{\mu\in\Gamma_{\mathrm{L}}^\vee/\Gamma_{\mathrm{L}}} \frac{\Theta_\mu^{\Gamma_{\mathrm{L}}}(\tau|p_\omega)}{\eta(\tau)^2}\frac{\bar\Theta^{\Gamma_R}_{\varphi(\mu)}(\bar\tau|0)}{\bar\eta(\bar\tau)^2}\left(\frac{\vartheta_1(\tau|z)}{\eta(\tau)}\right)^{r-2}\times$$

$$\left\{\frac{N}{12}\,\phi_{0,1}(\tau,z) + \left(-\frac{N-24}{12}\,\hat E_2(\tau) + f(p_{\mathrm{L}},\omega)\right)\phi_{-2,1}(\tau,z)\right\}$$

- $f(p_{\mathrm{L}},\omega) = \frac{1}{2}\int_{\mathcal{S}}\langle p_{\mathrm{L}},p_\omega\rangle^2_{\Gamma_{\mathrm{L}}} - \frac{N-24}{4\pi\tau_2}$

Threshold corrections

- New supersymmetric index:

$$Z_{\mathrm{NEW}}(\tau,\bar\tau) = \frac{1}{\eta(\tau)^2}\mathrm{Tr}_{\mathrm{R}}\left[\bar J_0 (-1)^{F_R} q^{L_0 - c/24} \bar q^{\bar L_0 - \bar c/24}\right]$$

$$= \frac{\bar\eta^2 E_4(\tau,0)}{2\eta^{10}} \sum_{\gamma,\delta=0}^{1} q^{\gamma^2} \left\{ \left(\frac{\vartheta_1\left(\tau \left| \frac{\gamma\tau+\delta}{2}\right.\right)}{\eta(\tau)} \right)^{8-r} Z_{\mathrm{FY}}\left(\tau,\bar\tau,\frac{\gamma\tau+\delta}{2}\right) \right\}$$

- In terms of standard almost holomorphic weak Jacobi forms:

$$Z_{\mathrm{NEW}}(\tau,\bar\tau) = \sum_{\mu\in\Gamma_{\mathrm{L}}^{\vee}/\Gamma_{\mathrm{L}}} \sum_{\substack{p_{\mathrm{L}}\in\Gamma_{\mathrm{L}}+\mu \\ p_{\mathrm{R}}\in\Gamma_{\mathrm{R}}+\varphi(\mu)}} q^{\frac12\langle p_{\mathrm{L}},p_{\mathrm{L}}\rangle_{\Gamma_{\mathrm{L}}}} \bar q^{\frac12\langle p_{\mathrm{R}},p_{\mathrm{R}}\rangle_{\Gamma_{\mathrm{R}}}} \times$$

$$\times \left(-\frac{N}{12}\frac{E_4E_6}{\Delta} + \frac{N-24}{12}\frac{E_4^2\hat E_2}{\Delta} - \frac{f(p_{\mathrm{L}},\omega)}{2}\frac{E_4^2}{\Delta} \right)$$

Threshold corrections

- Gauge threshold to E_8:

$$\Delta = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\,\tau_2\,\frac{1}{\eta(\tau)^2}\mathrm{Tr}_{\mathrm{R}}\left[\left(Q_{E_8}^2 - \frac{1}{8\pi\tau_2}\right)\bar{J}_0(-1)^{F_R} q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - \bar{c}/24}\right]$$

- $t := \frac{1}{2}\langle p_{\mathrm{L}}, p_{\mathrm{L}}\rangle_{\Gamma_{\mathrm{L}}}$

$$\begin{aligned}\Delta = \sum_{\mathrm{BPS}} &\left\{1 + \frac{N-24}{12t} + (t-1)\log\left(\frac{t-1}{t}\right)\right.\\ &\left. + \frac{\int_{\mathcal{S}}\langle p_{\mathrm{L}}, p_{\omega}\rangle^2}{4}\left[1 - \frac{1}{6t^2} - \frac{1}{2t} + (t-1)\log\left(\frac{t-1}{t}\right)\right]\right\}\\ &- 2(N+12)\mathcal{I}\end{aligned}$$

Mathieu moonshine?

- Mathieu moonshine? Natural decomposition into $N = 4$ Virasoro characters
- Mathieu moonshine in $K3 \times T^2$:

$$Z_{\rm ell}^{K3} = 24\,\mathrm{ch}_{h=1/4,l=0} + \sum_{n=0}^{\infty} A_n\,\mathrm{ch}_{h=k+1/4,l=1/2}$$

- Fu-Yau:

$$\frac{N}{12}\,\phi_{0,1}(\tau,z) + \left(-\frac{N-24}{12}\,\hat{E}_2(\tau) + f(p_{\rm L},\omega)\right)\phi_{-2,1}(\tau,z) =$$

$$N\,\mathrm{ch}_{h=1/4,l=0} + \sum_{n=0}^{\infty} \tilde{A}_n\,\mathrm{ch}_{h=n+1/4,l=1/2}$$

with $\tilde{A}_n = 2(8n-1)C_n + N\,\frac{A_n - 2(8n-1)C_n}{24} - \frac{1}{2}\int_{\mathcal{S}} \langle p_{\rm L}, p_\omega\rangle_{\Gamma_{\rm L}}^2\, C_n$

- C_n related to tri-partition of integers

Conclusion

- Corrections to the gravitational and gauge couplings as integrals of descendents of the new supersymmetric index over $SL_2(\mathbb{Z})$ fundamental domain. Unfold integration domain against Niebur-Poincaré series (Angelantonj, Florakis, Pioline (1203.0566))
- Comparison with prepotential computations for dual candidates on the type IIA side, Melnikov, Minasian, Theisen (1206.1417)
- Result for the index looks like a twining partition function in the context of Mathieu moonshine

n	$\tilde{A}_n$
0	$-2 - m(p_{\mathrm{L}})$
1	$42 + 2N - 3\,m(p_{\mathrm{L}})$
2	$270 + 8N - 9\,m(p_{\mathrm{L}})$
3	$1012 + 22N - 22\,m(p_{\mathrm{L}})$
4	$3162 + 58N - 51\,m(p_{\mathrm{L}})$
5	$8424 + 132N - 108\,m(p_{\mathrm{L}})$
6	$20774 + 294N - 221\,m(p_{\mathrm{L}})$
7	$47190 + 604N - 429\,m(p_{\mathrm{L}})$
8	$102060 + 1210N - 810\,m(p_{\mathrm{L}})$
9	$210018 + 2318N - 1479\,m(p_{\mathrm{L}})$
10	$417120 + 4334N - 2640\,m(p_{\mathrm{L}})$