Localization of $\mathcal{N} = (0, 2)$ GLSMs on the 'Coulomb branch'

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Supersymmetric gauge theories in two dimensions

Two-dimensional supersymmetric gauge theories—a.k.a. GLSM—are an interesting playground for the quantum field theorist.

- They exhibit many of the qualitative behaviors of their higher-dimensional cousins.
  - also useful to describe surface operators in 4d
- Supersymmetry allows us to perform exact computations.
- They provide useful UV completions of non-linear σ-models, including conformal ones, and of other interesting 2d SCFTs.
- Consequently, they are useful tools for string theory and enumerative geometry:
  - $\mathcal{N} = (2, 2)$ susy: type II string theory compactifications.
  - $\mathcal{N} = (0, 2)$ susy: heterotic compactifications.

GLSM Observables

Consider a GLSM with at least one $U(1)$ factor. We have the complexified FI parameter

$$\tau = \frac{\theta}{2\pi} + i\xi$$

which is classically marginal in 2d. $\xi \gg 1$ is a large volume limit.

Schematically, expectation values of appropriately supersymmetric local operators $\mathcal{O}$ have the expansion

$$\langle \mathcal{O} \rangle \sim \sum_k q^k Z_k(\mathcal{O}) \; , \qquad q = e^{2\pi i \tau} \; .$$

The 2d instantons are *gauge vortices*.

GLSM supersymmetric observables: the $(2,2)$ case

For theories with $\mathcal{N} = (2,2)$ supersymmetry, we can consider the half-BPS operators:

- $[\tilde{Q}_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$ (chiral ring)
- $[Q_-, \mathcal{O}] = [\tilde{Q}_+, \mathcal{O}] = 0$ (twisted chiral ring)

These operators have non-singular OPE:

$$\mathcal{O}_a \mathcal{O}_b \sim C_{ab}{}^c \mathcal{O}_c$$

The corresponding chiral rings are captured by TFTs:

$$\langle \mathcal{O}_a \, \mathcal{O}_b \cdots \rangle_{\Sigma_g}$$

defined by a topological twist of the ‘physical’ theory [Witten, 1988]:

- chiral ring $\leftrightarrow$ B-twist
- twisted chiral ring $\leftrightarrow$ A-twist

GLSM supersymmetric observables: the $(2,2)$ case

If we consider 'ordinary' gauge theories of vector and chiral multiplets:

$$\mathcal{V}^{(2,2)} = (a_\mu, \sigma, \tilde{\sigma}, \lambda, \tilde{\lambda}, D) \; , \qquad \Phi^{(2,2)} = (\phi, \tilde{\phi}, \psi, \tilde{\psi}, F, \tilde{F})$$

the simplest chiral and twisted chiral ring operators are holomorphic polynomials in ϕ and σ, respectively:

$$\mathcal{O}^{cc} = P(\phi) \; , \qquad \mathcal{O}^{ac} = P(\sigma)$$

This is far from the full story, but enough for our purpose.
We will focus on the twisted chiral operators:

$$\mathrm{Tr}(\sigma^p) \; , \qquad p = 0, 1, 2, \cdots$$

For theories that flow to a NLSM onto a Kähler manifold X, these operators flow to cohomology classes $H^{p,p}(X)$ in the IR.

GLSM supersymmetric observables: $(2, 2)$ case

In particular, for X_3 a CY threefold, the genus-zero correlators

$$\langle \mathrm{Tr}_I(\sigma)\, \mathrm{Tr}_J(\sigma)\, \mathrm{Tr}_K(\sigma) \rangle_{\mathbb{C}P^1} = Y_{IJK}(q)\ , \qquad I, J, K = 1, \cdots, h^{1,1}(X_3)$$

compute the holomorphic Yukawa couplings of the type II compactification on X_3, albeit in the *algebraic coordinates* $q = z$. They can often be computed using mirror symmetry.

For non-abelian GLSMs, we generally have more independent correlators of the form:

$$\langle \mathrm{Tr}(\sigma^{p_1})\, \mathrm{Tr}(\sigma^{p_2}) \cdots \rangle$$

They can be computed by localization.

[Morrison, Plesser, 1994]

[Szenes, Vergne, 2003]

[CC, Cremonesi, Park, 2015]

$\mathcal{N} = (0, 2)$ observables

A priori, the above would not generalize to $(0, 2)$ theories, which only have two right-moving supercharges with

$$Q_+^2 = 0 \,, \qquad \tilde{Q}_+^2 = 0 \qquad \{Q_+, \tilde{Q}_+\} = -4P_{\bar{z}} \,.$$

Half-BPS operators are $\tilde{Q}_+$-closed, and generally do not form a ring but a chiral algebra:

$$\mathcal{O}_a(z)\mathcal{O}_b(0) \sim \sum_c \frac{f_{abc}}{z^{s_a+s_b-s_c}} \mathcal{O}_c(z)$$

In some favorable cases with an extra $U(1)_L$ symmetry, there exists a subset of the $\mathcal{O}_a$, of spin $s = 0$, with trivial OPE. These pseudo-chiral rings are also known as “topological heterotic rings".

[Adams, Distler, Ernebjerg, 2006]

$\mathcal{N} = (0,2)$ localization: new result

In this talk, we will motivate a simple localization formula for some pseudo-chiral ring correlation functions in $(0,2)$ models with a "Coulomb branch"—-in particular, GLSMs with a "$(2,2)$ locus".

[CC, Gu, Jia, Sharpe, 2015]

For "Coulomb branch operators" similar to the $(2,2)$ case, we have:

$$\langle \mathcal{O}(\sigma) \rangle_{\mathbb{C}P^1} = \sum_k q^k \oint_{\text{JKG}} \frac{d\sigma}{2\pi i} Z_k^{\text{1-loop}}(\sigma)\, \mathcal{O}(\sigma)$$

for the so-called $A/2$-twist.

This can be generalized to correlators on Σ_g using recent results.

[CC, Kim, 2016]

Outline

$\mathcal{N} = (0, 2)$ models and quantum sheaf cohomology

Localizing $(0, 2)$ GLSMs with a $(2, 2)$ locus

Examples (abelian and non-abelian)

Conclusion and outlook

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$\mathcal{N}=(0,2)$ observables: Quantum sheaf cohomology

Consider a NLSM:

$$\Sigma \longrightarrow E$$

where E is an holomorphic vector bundle over the Kähler manifold X:

$$V \to E \to X \ .$$

The local coordinates on X are in chiral multiplets $\Phi_i = (\phi_i, \psi_i)$ and the local coordinates on the fiber V are in Fermi multiplets $\Lambda_I = (\Lambda_I, E_I)$.

The 'simplest' $\tilde{Q}_+$-closed operators are of the form:

$$\omega = \omega_{i_1 \cdots i_q I_1 \cdots I_p}(\phi, \tilde{\phi})\, \tilde{\psi}^{i_1} \cdots \tilde{\psi}^{i_q} \Lambda^{I_1} \cdots \Lambda^{I_p} \ , \qquad \bar{\partial}\omega = 0 \ .$$

They correspond to sheaf cohomology classes $H^q(X, \Lambda^p E^*)$.

We would like to "UV complete" this guys in a GLSM description.

Aside: Curved-space rigid supersymmetry

For massive theories, there is only one way to preserve supersymmetry on the sphere, unlike the $(2,2)$ case.

More precisely, assuming that we have a massive $\mathcal{N} = (0,2)$ theory (such as the GLSM) preserving $U(1)_R$, the theory has an $\mathcal{R}$-multiplet $\mathcal{R}_\mu = (j^{(R)}_\mu, S_\mu, T_{\mu\nu})$ [Dumitrescu, Seiberg, 2011] which couples to the metric and its superpartners in the usual way:

$$\mathscr{L} = \frac{1}{2}\Delta g_{\mu\nu} T^{\mu\nu} + A^{(R)}_\mu j^\mu_{(R)} + \Psi_\mu S^\mu \ .$$

It is easy to show that the *only* supersymmetric background à la [Festuccia, Seiberg, 2011] on Σ_g is the half-topological twist. [Witten, 1994] (In particular, there exists no $(0,2)$ Ω-background.)

This entails a choice of R-symmetry. Different choices can lead to 'twisted theories' with different properties.

$(0,2)$ GLSM with a $(2,2)$ locus and $A/2$-twist

We will focus on $(0,2)$ supersymmetric GLSMs with a $(2,2)$ locus. Schematically, they are determined by the following $(0,2)$ matter content:

- A vector multiplet $\mathcal{V}$ and a chiral Σ in the adjoint of the gauge group G, with $\mathfrak{g} = \mathrm{Lie}(G)$.
- Pairs of chiral and Fermi multiplets Φ_i and Λ_i, in representations $\mathfrak{R}_i$ of $\mathfrak{g}$.

The interactions are encoded in two sets of holomorphic functions of the chiral multiplets $\mathcal{E}_i$ and J_i.

We also turn on an FI term τ^I for each $U(1)_I$ in G.

$(0,2)$ GLSM with a $(2,2)$ locus and $A/2$-twist

We assign the R-charges:

$$R_{A/2}[\Sigma] = 0 \;, \qquad R_{A/2}[\Phi_i] = r_i \;, \qquad R_{A/2}[\Lambda_i] = r_i - 1 \;,$$

which is always anomaly-free.

We can define the theory on a curved two-manifold Σ_g by an half-twist:

$$S = S_0 + \frac{1}{2} R_{A/2} \;,$$

preserving one supercharge $\tilde{\mathcal{Q}} \sim \tilde{Q}_+$.
The R-charges r_i must be integers (typically, $r_i = 0$ or 2).

$(0,2)$ GLSM with a $(2,2)$ locus and $A/2$-twist

As a further assumption, we preserve a certain additional flavor $U(1)_L$ symmetry classically.

It is convenient to write it as $L = R_{\rm ax} - R_{A/2}$ with $R_{\rm ax}$ the 'axial' R-symmetry:

$$R_{\rm ax}[\sigma] = 2\,, \qquad R_{\rm ax}[\phi_i] = 0\,, \qquad R_{\rm ax}[\Lambda_i] = 1\,.$$

This is generally anomalous except for theories that flow to CFTs. In any case, it constrains us to choose a potential $\mathcal{E}_i$ linear in Σ and a potential J_i independent of Σ:

$$\mathcal{E}_i(\Sigma, \Phi) = \Sigma E_i(\Phi)\,, \qquad J_i = J_i(\Phi)$$

The Coulomb branch of theories with a $(2,2)$ locus

It is very useful to study the classical "Coulomb branch" spanned by the scalar σ in Σ:

$$\sigma = \mathrm{diag}(\sigma_a) \,, \qquad a = 1, \cdots, \mathrm{rank}(\mathfrak{g})$$

The matter fields obtain a mass

$$M_{ij} = \partial_j \mathcal{E}_i \big|_{\phi=0} = \sigma_a \, \partial_j E_i^a \big|_{\phi=0} \,.$$

By gauge invariance, M_{ij} is block-diagonal, with each block spanned by fields with the same gauge charges. We denote these blocks by M_γ.

Note: On the $(2,2)$ locus, $M_{ij} = \delta_{ij} Q_i(\sigma)$.

Let us denote by $\tilde{\mathfrak{M}} \cong \mathfrak{h}_{\mathbb{C}} \cong \mathbb{C}^{\mathrm{rank}(G)}$ the covering space of the classical Coulomb branch, spanned by $\{\sigma_a\}$.

The Coulomb branch and $J_{\rm eff}$

Since the matter fields are massive, we can integrate them out to obtain an effective theory on the "Coulomb branch".

Recall that the field strength $f_{\mu\nu}$ and the gaugini sit in a Fermi multiplet $\mathcal{Y}$ with the associated holomorphic potentials:

$$\mathcal{E}_{\mathcal{Y}} = 0\,, \qquad\qquad J_{\mathcal{Y}} = \tau\,,$$

If we integrate out the matter fields at a generic point on the Coulomb branch, we obtain the effective couplings:

$$(J^{\rm eff}_{\mathcal{Y}})_a = \tau^a - \frac{1}{2\pi i}\sum_\gamma \sum_{\rho_\gamma\in\mathfrak{R}_\gamma} \rho^a_\gamma \log\left(\det M_{(\gamma,\,\rho_\gamma)}\right) - \frac{1}{2}\sum_{\alpha>0}\alpha^a$$

[McOrist, Melnikov, 2007]

This also encodes the RG running of τ.

Pseudo-chiral ring relations from $J_{\rm eff}$

In analogy with the discussion of the twisted chiral ring of $\mathcal{N}=(2,2)$ theories, let us call the equations:

$$(J^{\rm eff}_{\mathcal{Y}})_a(\sigma) = 0 \;, \qquad\qquad \alpha(\sigma) \neq 0$$

the "Bethe equations" of the $(0,2)$ GLSM defined above. Note that we impose that any solution $\hat\sigma = \{\hat\sigma_a\}$ should be away from the walls of the Weyl chambers in $\tilde{\mathfrak{M}}$.

We expect that the 'Coulomb branch' operators $\mathrm{Tr}(\sigma^p)$ form a pseudo-chiral ring. Their algebra is encoded in $J_{\rm eff}$ according to:

$$\mathcal{A} = \mathbb{C}[\sigma_a]^{W_G}/I_{\rm BE}$$

where $I_{\rm BE}$ is the ideal generated by the relations satisfied by the solutions to the Bethe equations. (We will show this in a moment.)

Sheaf cohomology relations from J_{eff}

For GLSMs that flow to a NLSM over X, we have the well-motivated *conjecture:*

$$\mathrm{Tr}(\sigma^p) \qquad \longrightarrow \qquad \omega \in H^p(X, \Lambda^p E^*)$$

In the theories we are considering, E is a deformation of the holomorphic tangent bundle TX.

Then the ring $\mathcal{A}$ defined above is a sub-ring of the full (conjectured) quantum sheaf cohomology ring.

In simple-enough cases, it *is* the full QSC. For instance:

- Toric varieties
- Grassmannian manifold, flag manifolds

with E a deformation of TX are "simple enough" in that sense.

see e.g. [Donagi, Guffin, Katz, Sharpe, 2011]

Localization on the Coulomb branch

We would like to compute

$$\langle \mathcal{O}(\sigma) \rangle^{A/2}_{\mathbb{C}P^1}$$

in a way similar to recent computations of the elliptic genus [Benini, Eager, Hori, Tachikawa, 2013] and of A-twisted correlators on $\mathbb{C}P^1$ [CC, Cremonesi, Park, 2015] for GLSMs.

We use:

$$\langle \mathcal{O}(\sigma) \rangle^{A/2}_{\mathbb{C}P^1} = \langle \mathcal{O}(\sigma)\, e^{-S_{\mathrm{loc}}} \rangle^{A/2}_{\mathbb{C}P^1}$$

with

$$S_{\mathrm{loc}} = \frac{1}{e^2}(S_{YM} + S_{\Sigma}) + \frac{1}{g^2}\sum_i (S_{\Phi_i} + S_{\Lambda_i}) = \tilde{\mathcal{Q}}(\cdots)$$

and take the $e, g \to 0$ limit.

Localization on the Coulomb branch

The path integral localizes onto the 'zero-modes' of the vector multiplet on $\mathbb{C}P^1$:

$$\mathcal{V}_0 = (\tilde{\lambda}, \hat{D}) \,, \qquad \Sigma_0 = (\sigma, \tilde{\sigma}, \tilde{\psi}_\sigma)$$

They are constant modes on the sphere with the $A/2$-twist.
(In particular, $\tilde{\lambda}$ and $\tilde{\psi}_\sigma$ have twisted spin $s = 0$.)

We can go onto the classical Coulomb branch:

$$\sigma = \mathrm{diag}(\sigma_a) \,, \qquad a = 1, \cdots, \mathrm{rank}(G)$$

Diagonalizing the full vector multiplet leads to a sum over GNO-quantized fluxes: [Blau, Thompson, 1994]

$$\frac{1}{2\pi} \int_{\mathbb{C}P^1} da = k \in \Gamma_{G^\vee}$$

Localization on the Coulomb branch

The matter fields are massive at generic values of the 'background' $\mathcal{V}_0, \Sigma_0$, and we can integrate them out:

$$\langle \mathcal{O}(\sigma) \rangle^{A/2}_{\mathbb{C}P^1} = \sum_k q^k \int [d\mathcal{V}_0 \, d\Sigma_0] \; \mathcal{Z}_k(\mathcal{V}_0, \Sigma_0) \; \mathcal{O}(\sigma)$$

Here $\mathcal{Z}_k(\mathcal{V}_0, \Sigma_0)$ is a superdeterminant which one can compute in various ways.

The integration over fermionic zero modes $\tilde{\lambda}, \tilde{\psi}_\sigma$ has to be done carefully, but fortunately we can follow previous literature.

Supersymmetry is of great help:

$$\delta \mathcal{Z}_k = \left(\hat{D} \partial_{\tilde{\lambda}} + \tilde{\psi}_\sigma \partial_{\tilde{\sigma}} \right) \mathcal{Z}_k = 0$$

This helps convert the integral over the classical Coulomb branch into a *contour integral.*

A residue formula for $A/2$-model correlators on S^2

In this way, one can argue that our $A/2$-twisted correlators on S^2 are given by:

$$\langle \mathcal{O}(\sigma) \rangle^{A/2}_{\mathbb{C}P^1} = \frac{1}{|W_G|} \sum_k \oint_{\text{JKG}} \prod_{a=1}^{\text{rank}(G)} \left[d\sigma_a \, q_a^{k_a} \right] Z_k^{\text{1-loop}}(\sigma) \, \mathcal{O}(\sigma)$$

with

$$Z_k^{\text{1-loop}}(\sigma) = (-1)^{\sum_{\alpha>0}(\alpha(k)+1)} \prod_{\alpha>0} \alpha(\sigma)^2 \prod_\gamma \prod_{\rho_\gamma \in \mathfrak{R}_\gamma} \left(\det M_{(\gamma,\, \rho_\gamma)} \right)^{r_\gamma - 1 - \rho_\gamma(k)}$$

Here we have a new residue prescription generalizing the Jeffrey-Kirwan residue relevant on the $(2,2)$ locus.

The Jeffrey-Kirwan-Grothendieck residue

In the $(2,2)$ case, the Jeffrey-Kirwan residue determines a way to pick a middle-dimensional contour in

$$\mathbb{C}^r - \cup_{i\in I} H_i \;, \qquad I = \{i_1, \cdots, i_s\}\; (s \geq r) \qquad H_i = \{\sigma_a \,|\, Q_i(\sigma) = 0\} \;,$$

when the integrand has poles on H_i only. (Here $r = \mathrm{rank}(G)$.)

For generic $(0,2)$ deformations, we have an integrand with singularities on more general divisors of $\tilde{\mathfrak{M}} \cong \mathbb{C}^r$:

$$D_\gamma = \{\sigma_a \,|\, P_\gamma(\sigma) = 0\} \;,$$

which intersect at the origin only.

We introduced the notation

$$P_\gamma(\sigma) = \det M_\gamma \;\in \mathbb{C}[\sigma_1, \cdots, \sigma_r] \;, \qquad\qquad (r = \mathrm{rank}(G))$$

which is a homogeneous polynomial of degree $n_\gamma \geq 1$ in σ.

The Jeffrey-Kirwan-Grothendieck residue

To define the relevant Jeffrey-Kirwan-Grothendieck (JKG) residue, we introduce the data $\mathbf{P} = \{P_\gamma\}$ and $\mathbf{Q} = \{Q_\gamma\}$ of divisors D_γ and associated gauge charges Q_γ. The residue is defined by its action on the holomorphic forms:

$$\omega_S = d\sigma_1 \wedge \cdots \wedge d\sigma_r \, P_0 \prod_{b \in S} \frac{1}{P_b} \, ,$$

with $S = \{\gamma_1, \cdots, \gamma_r\}$, which is

$$\text{JKG-Res}[\eta] \; \omega_S = \begin{cases} \operatorname{sign}\left(\det(Q_S)\right) \, \operatorname{Res}_{(0)} \omega_S & \text{if} \quad \eta \in \text{Cone}(Q_S) \, , \\ 0 & \text{if} \quad \eta \notin \text{Cone}(Q_S) \end{cases}$$

with $\operatorname{Res}_{(0)}$ the (local) Grothendieck residue at the origin.

The Jeffrey-Kirwan-Grothendieck residue

The Grothendieck residue itself is defined as:

$$\mathrm{Res}_{(0)}\, \omega_S = \frac{1}{(2\pi i)^r} \oint_{\Gamma_\varepsilon} d\sigma_1 \wedge \cdots \wedge d\sigma_r \, \frac{P_0}{P_{\gamma_1} \cdots P_{\gamma_r}}$$

with the real r-dimensional contour:

$$\Gamma_\varepsilon = \left\{ \sigma \in \mathbb{C}^r \,\Big|\, |P_{\gamma_1}| = \varepsilon_1 \,, \cdots , |P_{\gamma_r}| = \varepsilon_r \right\}$$

and it is eminently computable.

Finally, we should take $\eta = \xi_{\mathrm{eff}}^{\mathrm{UV}}$ to cancel the “boundary contributions” from infinity on the Coulomb branch.

We really made two conjectures here: (1) The JKG actually exists as a local residue with nice properties. (2) It is the correct contour integral chosen by the path integral localization. (Full proof for $U(1)$ case only.)

Generalization to Σ_g

Following recent work [CC, Kim, 2016; Benini, Zaffaroni, 2016], we can easily generalize the above to a closed orientable Riemann surface Σ_g:

$$\langle \mathcal{O}(\sigma) \rangle_{\Sigma_g}^{A/2} = \frac{1}{|W|} \sum_k \oint_{\text{JKG}} \prod_{a=1}^{\text{rank}(G)} \left[d\sigma_a \, q_a^{k_a} \right] Z_{g,k}^{\text{1-loop}}(\sigma) \, H(\sigma)^g \, \mathcal{O}(\sigma)$$

with

$$Z_{g,k}^{\text{1-loop}}(\sigma) = (-1)^{\sum_{\alpha>0}(\alpha(k)+1)} \prod_{\alpha>0} \alpha(\sigma)^{2(1-g)}$$

$$\times \prod_{\gamma} \prod_{\rho_\gamma \in \mathfrak{R}_\gamma} \left(\det M_{(\gamma,\, \rho_\gamma)} \right)^{-(g-1)(r_\gamma - 1) - \rho_\gamma(k)}$$

and

$$H(\sigma) = \det_{ab} \left(\partial_{\sigma_b} J_a^{\text{eff}} \right)$$

Relation to the Bethe equations

By performing the sum over topological sectors (at least formally), one can rewrite the above result as:

$$\langle \mathcal{O}(\sigma) \rangle^{A/2}_{\Sigma_g} = \sum_{\hat\sigma \in \mathcal{S}_{\rm BE}} \mathcal{H}(\hat\sigma)^{g-1}\, \mathcal{O}(\hat\sigma)$$

with

$$\mathcal{H}(\sigma) = \left(Z^{\text{1-loop}}_{0,0}(\sigma) \right)^{-1} H(\sigma)$$

the 'handle-gluing operator'. The sum is over distinct solutions to the 'Bethe equations'

$$(J^{\rm eff}_{\mathcal{Y}})_a(\sigma) = 0\;, \qquad\qquad \alpha(\sigma) \neq 0$$

of the $(0,2)$ GLSM with a $(2,2)$ locus.

In the case when G is abelian and $r_\gamma = 0$ for all chiral multiplets, this reproduces previous results [McOrist, Melnikov, 2007].

Pseudo-chiral ring relations

The formula

$$\langle \mathcal{O}(\sigma) \rangle_{\Sigma_g}^{A/2} = \sum_{\hat\sigma \in \mathcal{S}_{\rm BE}} \mathcal{H}(\hat\sigma)^{g-1}\, \mathcal{O}(\hat\sigma)$$

makes it manifest that the correlation functions satisfy the pseudo-chiral ring relations defined above. We have

$$\langle \mathcal{O}(\sigma) f(\sigma) \rangle_{\Sigma_g}^{A/2} = 0$$

for any $f(\sigma)$ such that $f(\hat\sigma) = 0$—that is, for any pseudo-chiral relation.

This can also be seen from the integral representation of the correlators.

Further consequences of the localization formula

The explicit formula for the Coulomb branch correlators of the $A/2$-twisted GLSMs implies a few more results for the corresponding NLSMs.

- The correlators do not depend on the J_I potentials.
- The correlators only depend on the linear term in $\mathcal{E}_I \sim \sigma\phi_I + \cdots$. They do not depend on *non-linear E-deformations*.

These results were previously conjectured [McOrist, Melnikov, 2008]. They follow from our explicit result simply because we localized on $\Phi_i = 0$ for all the chiral multiplets.

Example: $\mathbb{C}P^1 \times \mathbb{C}P^1$ with deformed tangent bundle

Consider a theory with gauge group $U(1)^2$, two neutral chiral multiplets Σ_1, Σ_2 and four pairs of chiral and Fermi multiplets:

$$\Phi_i, \Lambda_i \ , \ i = 1,2 \quad Q_i = (1,0) \ , \qquad\qquad \Phi_j, \Lambda_j \ , \ j = 1,2 \quad Q_j = (0,1) \ ,$$

with holomorphic potentials $J_i = J_j = 0$ and

$$\mathcal{E}_i = \sigma_1 (A\phi)_i + \sigma_2 (B\phi)_i \ , \qquad \mathcal{E}_j = \sigma_1 (C\phi)_j + \sigma_2 (D\phi)_j \ .$$

with A, B, C, D arbitrary 2×2 constant matrices. This realizes a deformation of the tangent bundle to the holomorphic bundle $\mathbf{E}$ described by the cokernel:

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{\begin{pmatrix} A & B \\ C & D \end{pmatrix}} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathbf{E} \longrightarrow 0$$

$\mathbb{C}P^1 \times \mathbb{C}P^1$, continued.

We have two sets $\gamma = 1, 2$:

$$\det M_1 = \det(A\sigma_1 + B\sigma_2) \;, \qquad \det M_2 = \det(C\sigma_1 + D\sigma_2) \;.$$

The $g = 0$ Coulomb branch residue formula gives

$$\langle \sigma_1^{p_1} \sigma_2^{p_2} \rangle_{\mathbb{C}P^1}^{A/2} == \sum_{k_1, k_2 \in \mathbb{Z}} q_1^{k_1} q_2^{k_2} \oint_{\text{JKG}} d\sigma_1 d\sigma_2 \; \frac{\sigma_1^{p_1} \sigma_2^{p_2}}{(\det M_1)^{1+k_1} (\det M_2)^{1+k_2}}$$

This can be checked against independent mathematical computations of sheaf cohomology groups. [Anderson, Sharpe, unpublished]

This result also implies the "quantum sheaf cohomology relations":

$$\det M_1 = q_1 \;, \qquad \det M_2 = q_2 \;,$$

in the $A/2$-ring. [McOrist, Melnikov, 2007]

Example: The deformed Grassmannian

The 'simplest' non-abelian GLSM has $G = U(N_c)$ with:

- a chiral multiplet Σ in the adjoint
- and N_f pairs Φ_i, Λ_i in the fundamental representation of $U(N_c)$

We can turn on an FI parameter ξ for $U(1) \subset U(N_c)$. At $\xi \gg 0$, this model engineers the NLSM on the Grassmanian $Gr(N_c, N_f)$.

We consider $J_i = 0$ and the $\mathcal{E}$-potential:

$$\mathcal{E}_i = A_i^{\,j}\,\sigma\phi_j + \mathrm{Tr}(\sigma)\,B_i^{\,j}\phi_j \; .$$

We can set $A_i^{\,j} = \delta_j^i$ by a field redefinition.

If $1 < N_c < N_f - 1$, the tangent space $TGr(N_c, N_f)$ admits $N_f^2 - 1$ deformations. [Guo, Lu, Sharpe, 2016]
They are encoded in $B_i^{\,j}$ (modulo its trace).

The deformed Grassmannian, continued.

Thus we have the mass matrix:

$$M_a = \sigma_a A + \left(\sum_{b=1}^{N_c} \sigma_b\right) B~, \qquad\qquad a = 1, \cdots, N_c~,$$

on the Coulomb branch, and the $g = 0$ correlators:

$$\langle \mathcal{O}(\sigma) \rangle_{\mathbb{C}P^1}^{A/2} = \sum_{\mathbf{k}=0}^{\infty} q^{\mathbf{k}} \mathcal{Z}_{\mathbf{k}}$$

$$\mathcal{Z}_{\mathbf{k}} = \frac{(-1)^{(N_c-1)\mathbf{k}}}{N_c!} \sum_{k_a | \sum_a k_a = \mathbf{k}} \mathrm{Res}_{(0)} \frac{\prod_{a\neq b}(\sigma_a - \sigma_b)}{\prod_{a=1}^{N_c} (\det M_a)^{1+k_a}} \, \mathcal{O}(\sigma)\, d\sigma_1 \wedge \cdots \wedge d\sigma_{N_c}$$

The sum is over partitions of $\mathbf{k}$ into N_c non-negative integers.

One can easily check that the correlators satisfy the ring relations, which are the QSC relations in this case. See also [Guo, Lu, Sharpe, 2016]

Conclusions

- We studied $(0,2)$ supersymmetric gauge theories with a $(2,2)$ locus. The theories with a classical $R_{\rm ax}$ have a 'Coulomb branch', giving us extra milage.
- We found the $(0,2)$ generalization of a recent $(2,2)$ Coulomb branch formula for A-twisted correlation functions of Coulomb branch operators.
  - It involves an interesting JKG residue operation which deserves further study. In the simplest cases, it is just an ordinary Grothendieck residue.
  - The formula is very concrete and computationally powerful. It allows to study non-abelian GLSMs, which were previously out of reach.
- An analogous formula applies to $B/2$-twisted GLSMs related to the case considered here by a bundle dualization. [Sharpe, 2006]
- The "equivariant" deformation by masses for flavor symmetries is also straightforward.

What now?

The results we just discussed are only valid in a small corner of the vast world of $(0,2)$ gauge theories and observables.

What one would *really* want to do is:

- Compute pseudo-topological correlators in generic $(0,2)$ theories with a pseudo-chiral ring.
- Compute correlators of more general half-BPS operators in $(0,2)$ GLSMs—that is, understand the $(0,2)$ chiral algebra non-perturbatively.

Some very interesting results have been obtained already in the toric case, see esp. [McOrist, Melnikov, 2008]. To make further progress, one might need better methods to compute volumes of $(0,2)$ vortex moduli spaces.