Strominger system, Algebroids and Moduli

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$(0, 2)$ in Paris, IHP

Introduction

Candelas-Horowitz-Strominger-Witten (1985) :

Characterization of $N = 1$ supersymmetric Heterotic String compactifications with $M^{10} = M^4 \times X$, with constant dilaton and zero flux $H = 0$, by means of Calabi-Yau manifolds.

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- Moduli for non-Kahler heterotic compactifications?

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Recent developments on moduli of non-Kahler heterotic compactifications:

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- De la Ossa-Svanes / Anderson-Gray-Sharpe '14 : inf. moduli related with $H^1(\mathcal{Q})$ for holomorphic double extension

$$0 \to T^*X \to \mathcal{Q} \to \mathcal{E} \to 0$$
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$$0 \to \operatorname{End} V \oplus \operatorname{End} TX \to \mathcal{E} \to TX \to 0$$

Remark: Assume X is $\partial\overline{\partial}$ or $H^{0,2}(X) = 0$

Strominger - Hull geometry

- (X, Ω) complex 3-manifold: X complex with $\Omega \in \Omega^{3,0}_{hol}(X)$
- G: semi-simple compact Lie group
- $P_s \to X$: principal G-bundle

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Unknowns:

- hermitian metric g given by ω (where $\omega = g(J\cdot, \cdot)$),
- A connection (gauge field) on P_s, with curvature F (field strength)
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Taking into account the equations of motions: **Strominger system**

$$
\begin{aligned}
&F \wedge \omega^2 = 0, \qquad F^{0,2} = 0,\\
&R \wedge \omega^2 = 0, \qquad R^{0,2} = 0,\\
&d(\|\Omega\|_\omega \omega^2) = 0,\\
&dd^c\omega - \alpha'(\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F \wedge F) = 0
\end{aligned}
$$

Remark: as mathematicians, we cut the α' expansion at first order

Parameters and symmetries

M compact, oriented, $6d$ manifold, $P_s \to M$ a G-bundle. Consider

$$\mathcal{P} = \{(\Omega, \nabla, A, \omega) \in \Omega^3(\mathbb{C}) \times \text{affine connections} \times \text{ conn. on } P_s \times \Omega^2 \text{ satisfying } (1), (2), (3)\}$$

1. $\Omega \in \Omega^3(\mathbb{C})$ determines an almost complex structure J_Ω
2. ω is $J_\Omega -$ compatible
3. ∇ is a (ω, J_Ω)-unitary connection

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There is a natural groupoid of gauge transformations which acts on $\mathcal{P}$, preserving the solutions

$$\tilde{\mathcal{G}} = \{(g,p) \in \mathrm{Aut}(P_{Gl} \times_M P_s) \times \mathcal{P} \colon g^*(J_\Omega, \omega) = \check{g}^*(J_\Omega, \omega)\}$$

where $p = (\Omega, \nabla, A, \omega) \in \mathcal{P}$, P_{Gl} bundle of oriented frames and source/target

$$s(g,p) = p, \qquad t(g,p) = g^* p$$

Brute force

Given p a solution to ST:

$$p = (\Omega, \nabla, A, \omega) \in \mathcal{P}$$

consider the tangent space:

$$T_p\mathcal{P} \subset \Omega^{3,0+2,1} \oplus \Omega^1(M, \mathrm{End}\, TM) \oplus \Omega^1(M, \mathrm{ad}\, P_s) \oplus \Omega^2$$

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Infinitesimal action of the gauge groupoid Lie $\tilde{\mathcal{G}}$:

$$P \colon \Omega^0(TM) \oplus \Omega^0(\text{ad}\, P_s) \oplus \Omega^0(M, \text{End}_{\Omega,\omega}\, TM) \to T_p\mathcal{P}$$

$$P(V, \varphi, \psi) = (d\iota_{V^{1,0}}\Omega, \iota_V R_\nabla + d^\nabla(\nabla V) + \nabla\psi, \iota_V F_A + d_A\varphi, L_V\omega),$$

Brute force

Linearisation L of ST induces a complex of differential operators

$$0 \to \mathrm{Lie}\,\tilde{\mathcal{G}} \xrightarrow{P} T_p\mathcal{P} \xrightarrow{L} \Omega^4(\mathbb{C}) \oplus W \oplus \Omega^5 \oplus \Omega^4$$

with $W = \Omega^{(0,2)+6}(X, \mathrm{End}\, TX \oplus \mathrm{ad}\, P_s)$.

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(3d arrow $L = \oplus_{i=1}^5 L_i$)

$$\begin{aligned}
L_1(\dot\Omega, \dot\nabla, \dot a, \dot\omega) =& d\dot\Omega \\
L_2(\dot\Omega, \dot\nabla, \dot a, \dot\omega) =& (\overline{\partial} a^{0,1} + \frac{i}{2} F^{\dot J}, \overline{\partial}\dot\nabla^{0,1} + \frac{i}{2} R^{\dot J}) \\
L_3(\dot\Omega, \dot\nabla, \dot a, \dot\omega) =& (d_A \dot a \wedge \omega^2 + 2F \wedge \dot\omega \wedge \omega, d_\nabla \dot\nabla \wedge \omega^2 + 2R \wedge \dot\omega \wedge \omega) \\
L_4(\dot\Omega, \dot\nabla, \dot a, \dot\omega) =& d\left(2||\Omega_0||_{\omega_0}\dot\omega \wedge \omega_0 + (\dot{||\Omega||_\omega})\omega_0^2\right) \\
L_5(\dot\Omega, \dot\nabla, \dot a, \dot\omega) =& \frac{1}{2} d\left(J_0 d\dot\omega - J_0 (d\omega)^{\dot J J_0} + 4\alpha' \,\mathrm{tr}(\dot\nabla \wedge R) - 4\alpha' \,\mathrm{tr}(\dot a \wedge F)\right)
\end{aligned}$$

Space of infinitesimal deformations

Let S^* be the complex

$$0 \longrightarrow S^0 \xrightarrow{P} S^1 \xrightarrow{L} S^2$$

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Proposition (Garcia-Fernandez, Rubio, T.)

S^* is an elliptic complex

Definition:

The finite-dimensional space $H^1(S^*)$ is called space of infinitesimal deformations

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Many questions:

- Link with De la Ossa-Svanes / Anderson-Gray-Sharpe work?
- Decomposition in complex, metric and bundle moduli?
- Integration of infinitesimal deformations, obstructions?

Anomaly VS Bianchi

- The anomaly equation: local, Green-Schwarz mechanism,

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$$H = dB - \alpha'(CS(\nabla) - CS(A))$$

- Require *flux quantization:* the local B's glue into a closed 3-form flux, which defines an integral class in $H^3(M, \mathbb{Z})$.

Anomaly VS Bianchi

Recall

$$L_5 = \frac{1}{2} d \left(J_0 d\dot{\omega} - J_0 (d\omega)^{\dot{J} J_0} + 4\alpha' \operatorname{tr}(\dot{\nabla} \wedge R) - 4\alpha' \operatorname{tr}(\dot{a} \wedge F) \right).$$

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Define

$$d\mathcal{F} : H^1(S^*) \to H^3(M, \mathbb{R})$$

$$(\dot{\Omega}, \dot{\nabla}, \dot{a}, \dot{\omega}) \mapsto \frac{1}{2} \left(J_0 d\dot{\omega} - J_0 (d\omega)^{\dot{J} J_0} + 4\alpha' \operatorname{tr}(\dot{\nabla} \wedge R) - 4\alpha' \operatorname{tr}(\dot{a} \wedge F) \right).$$

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L_5 is the linearization of Bianchi identity,

$$dd^c\omega - \alpha'(\operatorname{tr} R \wedge R - \operatorname{tr} F \wedge F) = 0,$$

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$d\mathcal{F}$ is the linearization of anomaly equation. Set

$$H^1(\widetilde{S}^*) := \ker d\mathcal{F},$$

whose elements preserve flux quantization

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Substitute L_5 by $\widetilde{L}_5$ (*linearization of anomaly equation*)

$$\widetilde{L}_5(\dot{\Omega}, \dot{\nabla}, \dot{a}, \dot{\omega}, b) = db - \frac{1}{2}\left(J_0 d\dot{\omega} - J_0 (d\omega)^{\dot{J}J_0} + 4\alpha' \operatorname{tr}(\dot{\nabla} \wedge R) - \dots\right)$$

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meaning of this equation?

Strominger meets generalized geometry

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Infinitesimal symmetries:

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Elements of $\widetilde{S}^1$: infinitesimal variations of **generalized metrics** $V_+ \subset E$

Exact case $(T + T^*, \langle , \rangle, [,], \pi_T)$

Courant algebroid structure on $T + T^*$:

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(i_X\eta + i_Y\xi), \qquad [X + \xi, Y + \eta] = [X, Y] + L_X\eta - i_Y d\xi$$

It has structure group $\mathrm{O}(n, n)$, and symmetries include closed 2-forms, B-fields:

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Twisted version: an **exact** Courant algebroid $(E, \langle , \rangle, [,], \pi_T)$ (+ axioms)

$$0 \to T^* \to E \to T \to 0.$$

It is isomorphic, by choosing a (non-canonical) splitting, to

$$(T+T^*, \langle , \rangle, [,]_H := [,] + i_X i_Y H, \pi_T),$$

for some $H \in \Omega^3_{cl}(M)$ (whose class $[H] \in H^3(M)$ parameterizes E).

Transitive case $T^* \to E \to T \to 0$

Given a principal G-bundle P, we obtain by reduction a transitive Courant algebroid E:

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As a vector bundle,

$$E \cong T + \mathrm{ad}\, P + T^*$$

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Choosing a splitting $T \to E$, E is isomorphic to

$$(T + \mathrm{ad}\, P + T^*, \langle , \rangle, [,]_{\theta,H}, \pi_T),$$

where θ is a connection on P (with curvature $F_\theta \in \Omega^2_{cl}(\mathrm{ad}\, P)$), and $H \in \Omega^3(M)$ such that

$$dH - \langle F_\theta \wedge F_\theta \rangle = 0.$$

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Remark: This last equation is like the Bianchi identity: for any solution to ST, setting

$$P := P_s \times_M P_{Gl}$$

build a transitive Courant algebroid using the 3-form H and $F_\theta = F_A + R_\nabla$ the curvature of the product connection $\theta = A \times \nabla$. Choose the pairing so that:

$$\langle F_\theta \wedge F_\theta \rangle = \alpha' (\operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A)$$

The bracket

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$$
\begin{aligned}
[X + r + \xi, Y + t + \eta]_{\theta,H} = \\
&[X, Y] + L_X\eta - i_Y d\xi + i_Y i_X H \\
&- F_\theta(X, Y) + i_X dt - i_Y dr \\
&+ 2c(tdr) + 2c(i_X F_\theta t) - 2c(i_Y F_\theta r).
\end{aligned}
$$

Generalized metric for E exact:

A metric is a reduction of the frame bundle from $\mathrm{GL}(n)$ to $\mathrm{O}(n)$.

A generalized metric is a reduction from $\mathrm{O}(n, n)$ to $\mathrm{O}(n) \times \mathrm{O}(n)$.

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Equivalent to a rank n positive-definite subbundle

$$V_+ \subset E.$$

Since T^* is isotropic, $\pi : V_+ \to T$ is an isomorphism, so T inherits a positive-definite pairing, i.e., a usual metric g.

Generalized metric for E exact:

A metric is a reduction of the frame bundle from $\mathrm{GL}(n)$ to $\mathrm{O}(n)$.

A generalized metric is a reduction from $\mathrm{O}(n,n)$ to $\mathrm{O}(n)\times\mathrm{O}(n)$.

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generalized metric for E transitive:

For E transitive, the structure group is $\mathrm{O}(t, s)$,

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A connnection on E is a differential operator

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Generalized curvature and torsion are defined

The Gualtieri-Bismut connection

Let V_+ be an admissible generalized metric.
We have $V_- := (V_+)^\perp \cong T$ and $V_+ \cong E/T^*(\cong T + \mathrm{ad}\, P)$.

Let $C_+ \cong (\mathrm{ad}\, P)^\perp \subset T + \mathrm{ad}\, P$.
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$$D^B_e e' := [e_-, e'_+]_+ + [e_+, e'_-]_- + [Ce_-, e'_-]_- + [Ce_+, e'_+]_+,$$

The connection D_B preserves $V_\pm$ and has totally skew torsion T_{D_B}.

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Given $\varphi \in C^\infty(M)$, D^{LC} modified canonically to D^φ, compatible, torsion-free.

Generalized Killing spinor equations

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$$\not{D}^\varphi_- : S_+(V_-) \to S_-(V_-).$$

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Given a generalized metric V_+, as before, and $\varphi \in C^\infty(M)$, the *Killing spinor equations* for a spinor $\eta \in S_+(V_-)$ are given by

Killing spinor equations, Waldram-Strickland-Constable-Coimbra

$$D_+^\varphi \eta = 0,$$
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On a six-dimensinal spin-manifold

Theorem (Garcia-Fernandez, Rubio, T.)

Assume that E is exact. Then (V_+, φ, η) is a solution to the Killing spinor equations with $\eta \neq 0$ if and only if $H = 0$, φ is constant and g is a metric with holonomy contained in $SU(3)$.

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Theorem (Garcia-Fernandez, Rubio, T.)

Assume that E is transitive. The Strominger system is equivalent to the Killing spinor equations.

A couple of ideas from the proofs

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$$F_\theta \cdot \eta = 0$$
$$\nabla^- \eta = 0,$$
$$(H - 2d\varphi)\cdot\eta = 0,$$
$$dH - \langle F_\theta \wedge F_\theta \rangle = 0,$$

for $((g, H, \theta), \varphi, \eta)$, where, by $V_- \cong (T, g)$, $\eta \in S_+(T) \cong S_+(V_-)$ (and ∇^- is the Bismut connection with skew-torsion $-H$).

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For the converse in Strominger, given (ω, A, ∇), one defines $\theta = A \times \nabla$, $H = d^c\omega$ and φ. Note that the Bianchi identity

$$dd^c\omega - (\mathrm{tr}\, R \wedge R - \mathrm{tr}\, F_A \wedge F_A) = 0$$

corresponds to

$$dH - \langle F_\theta \wedge F_\theta \rangle = 0.$$

Infinitesimal generalized Killing spinors

Recall

$H^1(S^*) =$ **infinitesimal solutions to ST, modulo symmetries**

and the map (linearisation of anomaly equation):

$$d\mathcal{F} : H^1(S^*) \to H^3(M, \mathbb{R})$$

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$H^1(\widetilde{S}^*) =$ **infinitesimal solutions to ST, compatible with flux quantization, modulo symmetries of** E.

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Elements of $\Omega^0(E)$ induce inner symmetries of E. By restriction, we obtain an elliptic complex $\widehat{S}^*$ such that:

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is the differential of a *Flux map* $\mathcal{F}$:

$$\mathcal{F} : \mathcal{M}_{ST} \to H^3(X, \mathbb{R})$$

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$d\mathcal{F}$ is a closed $H^3(M, \mathbb{R})$-valued 1-form

$$d\mathcal{F} \in \Omega^1(\mathcal{M}_{ST}, H^3(M, \mathbb{R})),$$

and provides a foliation (integrating Ker $d\mathcal{F}$) on the moduli space

The flux map

The leaf of the foliation passing through p is the local moduli space

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obtain and $H^2(M,\mathbb{R})$-bundle $\widehat{\mathcal{M}}_{ST}$ over $\widetilde{\mathcal{M}}_{ST}$

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Conjecture

The moduli $\widehat{\mathcal{M}}_{ST}$ carries a natural Kähler structure.

Evidence: there is a natural map $T_p\widehat{\mathcal{M}}_{ST} \to H^1(\mathcal{Q})$ (complex).

AGSOS map

De la Ossa-Svanes/Anderson-Gray-Sharpe interpretation: cocycles in the Dolbeault complex of $\mathcal{Q}$:

$$0 \to T^*X \to \mathcal{Q} \to \mathcal{E} \to 0$$

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Fu-Yau example (elliptic fibration over $K3$): X is not $\partial\bar{\partial}$ and $h^{0,2}(X) = 1$.

Deformations of pairs

Elliptic complex encoding deformations of pairs of complex structures on M and $P_s \times_M P_{GL}$:

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Natural questions:

- Image of Ψ?
- Interpretation of $\operatorname{Ker}\Psi$ as a metric moduli?

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Recall the definition of Aeppli cohomology groups:

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for some vector field V and $r^c \in \Omega^0(\mathrm{ad}\, P)$.

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Injectivity of Φ would provide a splitting of infinitesimal moduli as holomorphic components and metric components.